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SOME RESULTS ON λ-DESIGN CONJECTURE

AJEET KUMAR YADAV

Abstract. Let v and λ be integers with $0 < \lambda < v$. A λ-design D is a pair $(X, \mathcal{A})$, where X is a finite set with v elements called points and $\mathcal{A}$ is a family of subsets of X called blocks, with $|\mathcal{A}| = |X|$ such that

(1) for all $B_i, B_j \in \mathcal{A}$, $i \neq j$, $|B_i \cap B_j| = \lambda$;
(2) for all $B_j \in \mathcal{A}$, $|B_j| = k_j > \lambda$, and not all k_j are equal.

The only known examples of λ-designs are so called of type-1 designs, which are obtained from symmetric designs by a certain complementation procedure. Ryser and Woodall had independently conjectured that all λ-designs are of type-1. Suppose r and $r^*(r > r^*)$ are replication numbers of D and for distinct points x and y of D, let $\lambda(x, y)$ denote the number of blocks of X containing x and y.

In this paper we investigate the possibilities of λ-designs to be of type-1 under the condition that $|\lambda(x, y) - \lambda(x, y')| < 2\left(\frac{r - r^*}{r + r^* - 2}\right)$. Under this condition, we prove that if $\frac{r-1}{r^*-1} \leq 3$, then λ-design D is of type-1. Also we prove that D has exactly two distinct block sizes.

1. Introduction

Let v, k and λ be positive integers with $\lambda < k < v$. A (v, k, λ) symmetric block design is a pair $(X, \mathcal{A})$, where X is a finite set with v elements, called points and $\mathcal{A}$ is a finite family of

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*Corresponding author.

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subsets of X, called blocks with $|\mathcal{A}| = |X|$, such that each block contains k points and each pair of points occurs in λ blocks.

A λ-design is a pair $(X, \mathcal{A})$, where X is a finite set with v points and $\mathcal{A}$ be a family of subsets of X with $|\mathcal{A}| = |X|$ such that

(1) $|B_i \cap B_j| = \lambda$ for all $B_i, B_j \in \mathcal{A}$, $i \neq j$;
(2) $|B_j| = k_j > \lambda$, for all $B_j \in \mathcal{A}$, and not all k_j are equal.

All known examples of λ-designs can be described by the following procedure: Let (X, ξ) be a symmetric $(v, k, k - \lambda)$ design with block set $\xi = \{A_1, A_2, \ldots, A_v\}$. Then

$$\mathcal{A} = \{B \subset X : B = A_1 \text{ or } B = (A_i \setminus A_1) \cup (A_1 \setminus A_i) \text{ for some } 2 \leq i \leq v\},$$

then $D = (X, \mathcal{A})$ is a λ-design. This construction is called complementation with respect to the block A_1 and λ-designs obtained by this procedure are called of type-1 λ-designs. The λ-design conjecture, also known as *Ryser-Woodall conjecture* due to Ryser [15] and Woodall [22], states that all λ-designs are of type-1.

Fiala [5] proposed the following three weaker conjectures than the λ-design conjecture:

(1) Conjecture 1: Every λ-design has at least one block of size 2λ.
(2) Conjecture 2: All λ-designs have exactly two block sizes.
(3) Conjecture 3: All λ-designs with two block sizes, one of which is 2λ, are of type-1.

Conjecture 3 is proved by T. D. Parulekar, S. S. Sane [14].

Stanton [20] used the term *Ryser-design* for *λ-design*, and some authors prefer to use the term *Ryser-design* for λ-design.

Let $D = (X, \mathcal{A})$ be a λ-design of index λ with $|X| = v$, for any $x \in X$, the number of elements of $\mathcal{A}$ containing x is called the replication number r or $r(x)$ of x in $\mathcal{A}$.

Ryser [15] and Woodall [22] independently proved that there are distinct integers r and r^*, both greater than one, such that $r + r^* = v + 1$ and every point of D has replication number r or r^*. We take $E(D)$(or E) as the set of all points of D with replication number r, $E^*(D)$ (or E^*) as the set of points of D with replication number r^*, $e = |E(D)|$ and $e^* = |E^*(D)|, g = \gcd(r - 1, r^* - 1), m = \gcd((r - r^*)/g, \lambda)$. Let $\kappa_B = |B \cap E(D)|, \kappa^*_B = |B \cap E^*(D)|, e + e^* = v$ and $\kappa_B + \kappa^*_B = |B|$.

The Ryser-Woodall conjecture has been proved for $\lambda = 1$ by N.G. de Bruijn and P. Erdös [4], $\lambda = 2$ by Ryser [15], $\lambda = 3$ by Bridges and Kramer [2], $\lambda = 4$ by Bridges [3], $5 \leq \lambda \leq 9$ by Kramer [12, 13] and Weisz [21] for $\lambda \leq 34$, N. M. Singhi and S. S. Shrikhande [17] proved Ryser-Woodall conjecture for $\lambda = p$, and A. Seress [16] proved for $\lambda = 2p$, for all prime p.

Singhi, Shrikhande and Pawale [18] proved if for all block B_j of a λ-design D, $\kappa_B - \kappa_B^*$ is divisible by mg_1 (g_1 is the largest divisor of $\frac{\lambda}{m}$) then D is of type-1.

Y. J. Ionin and M. S. Shrikhande [9, 10] developed a technique to investigate the conjecture as a function of v. They proved that any λ-design with $g \in \{1,2,3,4\}$ is of type-1. As a consequence, they proved that the Ryser-Woodall conjecture is true for any Ryser-design of index λ on $p+1, 2p+1, 3p+1$, or $4p+1$ points, where p is a prime. Using the same technique, Hein and Ionin [8] proved that the Ryser-Woodall conjecture is true for $g = 5$ and $v = 5p+1$, where p is a prime not congruent to 2 or 8 (mod 15). Fiala [5, 6, 7] proved that any λ-design with $g \in \{6,7,8\}$ is of type-1. Fiala [5, 6] proved that Ryser-Woodall conjecture is true for any λ-design on $v = 6p+1$ points, where p is a prime, or $v = 8p+1$ points, where p is a prime not congruent to 5 or 11 (mod 24).

Alraqad and Shrikhande [1] proved that λ-design with two block sizes are of type-1 for $g = 9, 11, 12, 13, 15, 16, 17, 19, 20, 21$; for $g = 10, 14, 18, 22$ with $v \neq 4\lambda - 1$ and for $v = 9p+1, v = 12p+1$ with certain restrictions.

In [23], Yadav, Pawale and Shrikhande fixed the difference $r - r^* = \theta$ and obtained that there are finitely many type-2 λ-designs with replication numbers r^* and $r = r^* + \theta$. Also it was proved that λ-designs with replication numbers r^*, $r = r^* + 4p$ and $r^* \neq (p-1)^2$, or $v = 7p+1$ such that $p \not\equiv 1, 13 \pmod{21}$ and $p \not\equiv 4, 9, 19, 24 \pmod{35}$, where p is a positive prime, are of type-1. In [24], they have proved that λ-designs with $r - r^* = p^n$, where p is prime and n is a positive integer, are of of type-1. In [23] and [24] several inequalities involving parameters of λ-design with $\lambda(x,y)$ are obtained where equality holds for type-1 designs, where $\lambda(x,y)$ denotes the number of blocks containing points x and y.

It was obtained by Yadav, Pawale and Shrikhande in [23], that there exist integer $t(x,y)$ such that

$$\lambda(x,y) = \begin{cases} \dfrac{r(r-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{for} \quad r(x) = r(y) = r; \\ \dfrac{r^*(r^*-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{for} \quad r(x) = r(y) = r^*; \\ \dfrac{r(r^*-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{for} \quad r(x) \neq r(y), \end{cases}$$

where

$$m' = \begin{cases} m \text{ if } m \text{ is odd;} \\ m/2 \text{ if } m \text{ is even.} \end{cases}$$

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It was observed that if $|t(x,y)-t(x,y')| \leq m'$ for $x \in E$ and distinct y, y' in $E \setminus \{x\}$ or E^* then D is of type-1. In this article we investigate the possibilities of λ-designs to be of type-1 under the condition that $|t(x,y)-t(x,y')| \leq 2m'$, for $x \in E$ and distinct $y, y' \in E \setminus \{x\}$ or E^*. This is equivalent to $|\lambda(x,y)-\lambda(x,y')| < \dfrac{2(r-r^*)}{r+r^*-2} < 2$. We prove that if $\gcd((r-r^*)/g, \lambda) = m$ and $\rho = \frac{r-1}{r^*-1} \neq \frac{2m+k}{k}$, where $1 \leq k \leq m-1$, then D is of type-1. As a consequence we show that if $\gcd(r-r^*, r+r^*) = s$, where s is positive integer greater than two, with $\rho \neq \frac{2s+k}{k}$, where $1 \leq k \leq s-1$, then D is of type-1. We prove that if $\rho \leq 3$, then λ-design D is of type-1, using this we prove that if $\lambda \leq 3$, then D is of type-1. Hence we prove that D has exactly two distinct block sizes under our assumtion.

2. Preliminaries

This section contains results on λ-designs, needed in sections 3. Readers may refer to [11, chapter 14] for basics on λ-design.

Theorem 2.1. (1) *D is a λ-design of type-1, whenever $m = 1$ or 2.*
(2) *D is a λ-design of type-1, whenever $s = 1$ or 2.*

Proof. (1) Shrikhande and Singhi [19, Lemma 4.4], proved that a λ-design with $m = 1$ is of type-1, and Seress [16, Theorem 1.4], proved that with $m = 2$ is of type-1. Alternatively, see [11, Theorem 14.3.3].
(2) See [19]; or [11, Theorem 14.5.14].

□

Theorem 2.2. [23, Theorem 3.1] *If $m \geq 1$, then for distinct points $x, y \in X$, there exists an integer $t(x,y)$ such that*

$$\lambda(x,y) = \begin{cases} \dfrac{r(r-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{if } r(x) = r(y) = r, \\ \dfrac{r^*(r^*-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{if } r(x) = r(y) = r^*, \\ \dfrac{r(r^*-1)}{(v-1)} + \dfrac{(r-r^*)}{(v-1)m'}t(x,y) & \text{if } r(x) \neq r(y). \end{cases}$$

Theorem 2.3. [11, Theorem 14.1.17] *For any λ-design of index λ on v points with $\lambda \geq 2$ having replication numbers $r > r^*$,*

$$\frac{\lambda}{\lambda-1} \leq \rho \leq \lambda, \tag{2.1}$$

and $\rho \notin (\lambda-1, \lambda)$.

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Theorem 2.4. [11, Theorem 14.2.10] *Let $D = (X, \mathcal{A})$ be a λ-design with replication numbers r and r^*. For distinct points x and y of D, $\lambda(x, y)$ denotes the number of blocks containing points x and y. Suppose there exist integers μ, μ^*, and $\overline{\mu}$ such that for any distinct points $x, y \in X$,*

$$\lambda(x,y) = \begin{cases} \mu & \text{if } r(x) = r(y) = r, \\ \mu^* & \text{if } r(x) = r(y) = r^*, \\ \overline{\mu} & \text{if } r(x) \neq r(y). \end{cases}$$

Then D is of type-1.

3. Main Results

In view of the Theorem 2.2, we investigate λ-designs under the condition that $|t(x, y) - t(x, y')| \leq 2m'$ for $x \in E$ and distinct y, y' in $E \setminus \{x\}$ or E^*. This is equivalent to $|\lambda(x, y) - \lambda(x, y')| < \dfrac{2(r - r^*)}{r + r^* - 2} < 2$.

Theorem 3.1. *Let $x \in E$, and suppose $|t(x, y) - t(x, y')| \leq 2m'$ for distinct y, y' in $E \setminus \{x\}$ or E^*.*

1. *If $\rho \leq 3$, then D is of type-1.*
2. *If $\rho \neq \frac{2m'+k}{k}$, where $1 \leq k \leq m' - 1$, then D is of type-1.*

Proof. Let $x \in E$ and distinct y, y' in $X \setminus \{x\}$. From Theorem 2.2, we get

$$|\lambda(x,y) - \lambda(x,y')| = \frac{1}{m'}\left(\frac{r - r^*}{r + r^* - 2}\right)|t(x,y) - t(x,y')|. \tag{3.1}$$

Let $x \in E$, and suppose $|t(x, y) - t(x, y')| \leq 2m'$ for distinct y, y' in $X \setminus \{x\}$.

Possible values of $|t(x, y) - t(x, y')| = 0, 1, \ldots, 2m'$. If $|t(x, y) - t(x, y')| = 1, 2, \ldots, m'$ then the right hand side of the Equation (3.1) is not an integer but the left hand side is an integer, which is a contradiction. Suppose $|t(x, y) - t(x, y')| = m' + k$, $1 \leq k \leq m' - 1$. This implies

$$|\lambda(x,y) - \lambda(x,y')| \leq \frac{2m' - 1}{m'}\frac{r - r^*}{r + r^* - 2} < \frac{2(r - r^*)}{r + r^* - 2}. \tag{3.2}$$

Observe that $\rho < 3$ if and only if $\frac{2(r-r^*)}{r+r^*-2} < 1$. Hence $\rho < 3$ implies $\lambda(x, y)$ is uniquely determined. Therefore by the Theorem 2.4, D is of type-1.

If $\rho = 3$ then $s = \gcd(r - r^*, r + r^*) = \gcd(2r^* - 2, 4r^* - 2) = 2$. Hence by the Theorem 2.1, D is of type-1.

Also $1 \leq l := \frac{m'+k}{m'}\frac{r-r^*}{r+r^*-2} < 2$, hence $l = 1$, since left hand side of the Equation (3.1) is a positive integer, which gives $\frac{r-1}{r^*-1} = \frac{2m'+k}{k}$. Hence $\rho = \frac{2m'+k}{k}$, $1 \leq k \leq m' - 1$, which is a contradiction to our assumption.

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If $|t(x,y)-t(x,y')| = 2m'$, then the Equation (3.1) implies $\rho = 3$. As before D is of type-1.

If $|t(x,y)-t(x,y')| = 0$, then $\lambda(x,y)$ is uniquely determined. Therefore by the Theorem 2.4, D is of type-1.

□

By using similar techniques as used by Y. J. Ionin and M. S. Shrikhande [11, Theorem 14.2.10], we prove Fiala's second conjecture in the following result.

Theorem 3.2. *Let $D = (X, \mathcal{A})$ be a λ-design of index λ with replication numbers r and r^*. Then D has exactly two distinct block sizes.*

Proof. Let $x \in E$ and distinct y, y' in $X \setminus \{x\}$. From the Equation (3.2) we get,

$$|\lambda(x,y) - \lambda(x,y')| < \frac{2(r-r^*)}{r+r^*-2} < 2. \tag{3.3}$$

We assume that for some positive integers a, b and c:

$$\lambda(x,y) = \begin{cases} a \quad \textit{or} \quad a+1 & \text{if} \quad x,y \in E \quad \textit{and } x \neq y; \\ b \quad \textit{or} \quad b+1 & \text{if} \quad x,y \in E^* \textit{ and } x \neq y; \\ c \quad \textit{or} \quad c+1 & \text{if} \quad x \in E,\, y \in E^*. \end{cases}$$

Also $\lambda(x,y) = r$ if $y = x \in E$ and $\lambda(x,y) = r^*$ if $y = x \in E^*$. For any block A of D, let

$$L(A) = \sum_{x \in E} \sum_{y \in A} \lambda(x,y)$$

and

$$L^*(A) = \sum_{x \in E^*} \sum_{y \in A} \lambda(x,y).$$

For any $x \in E$, we take $\alpha_1 = |\{y \in A \cap E / x \in A, \lambda(x,y) = a\}|$ and
$\alpha_2 = |\{y \in A \cap E / x \notin A, \lambda(x,y) = a\}|$. Observe that

$$\sum_{y \in A \cap E} \lambda(x,y) = \begin{cases} \alpha_1 a + (\kappa_A - \alpha_1 - 1)(a+1) + r \text{ if } x \in A, \\ \alpha_2 a + (\kappa_A - \alpha_2)(a+1) \text{ if } x \notin A. \end{cases}$$

Take $\beta_1 = |\{y \in A \cap E^* / \lambda(x,y) = c\}|$ and observe that

$$\sum_{y \in A \cap E^*} \lambda(x,y) = \beta_1 c + (\kappa_A^* - \beta_1)(c+1).$$

$$\begin{aligned} L(A) &= \sum_{x\in E}\sum_{y\in A\cap E}\lambda(x,y)+\sum_{x\in E}\sum_{y\in A\cap E^*}\lambda(x,y). \\ &= \kappa_A(\alpha_1 a+(\kappa_A-\alpha_1-1)(a+1)+r)+(e-\kappa_A)(\alpha_2 a+(\kappa_A-\alpha_2)(a+1)) \\ &\quad +e(\beta_1 c+(\kappa_A^*-\beta_1)(c+1)) \\ &= \kappa_A(\kappa_A(a+1)-\alpha_1-a-1+r)+(e-\kappa_A)(\kappa_A(a+1)-\alpha_2) \\ &\quad +e(\kappa_A^*(c+1)-\beta_1) \\ &= \kappa_A(\alpha_2-\alpha_1-a-1+r)+e(\kappa_A a+\kappa_A^* c+|A|-\beta_1) \\ &= \kappa_A(\alpha_2-\alpha_1-a-1+r+ea)+e(\kappa_A^* c+|A|-\beta_1). \end{aligned}$$

For any $x \in E^*$, we take $\gamma_1 = |\{y \in A \cap E^*/x \in A, \lambda(x,y) = b\}|$ and $\gamma_2 = |\{y \in A \cap E^*/x \notin A, \lambda(x,y) = b\}|$. Observe that

$$\sum_{y\in A\cap E^*}\lambda(x,y)=\begin{cases}\gamma_1 b+(\kappa_A^*-\gamma_1-1)(b+1)+r^* \text{ if } x\in A,\\ \gamma_2 b+(\kappa_A^*-\gamma_2)(b+1) \text{ if } x\notin A.\end{cases}$$

Take $\beta_2 = |\{y \in A \cap E/\lambda(x,y) = c\}|$ and observe that

$$\sum_{y\in A\cap E}\lambda(x,y)=\beta_2 c+(\kappa_A-\beta_2)(c+1).$$

$$\begin{aligned} L^*(A) &= \sum_{x\in E^*}\sum_{y\in A\cap E^*}\lambda(x,y)+\sum_{x\in E^*}\sum_{y\in A\cap E}\lambda(x,y). \\ &= \kappa_A^*(\gamma_1 b+(\kappa_A^*-\gamma_1-1)(b+1)+r^*)+(e^*-\kappa_A^*)(\gamma_2 b+(\kappa_A^*-\gamma_2)(b+1)) \\ &\quad +e^*(\beta_2 c+(\kappa_A-\beta_2)(c+1)) \\ &= \kappa_A^*(\kappa_A^*(b+1)-\gamma_1-b-1+r^*)+(e^*-\kappa_A^*)(\kappa_A^*(b+1)-\gamma_2) \\ &\quad +e(\kappa_A(c+1)-\beta_2) \\ &= \kappa_A^*(\gamma_2-\gamma_1-b-1+r^*)+e^*(\kappa_A^* b+\kappa_A c+|A|-\beta_2) \\ &= \kappa_A^*(\gamma_2-\gamma_1-b-1+r^*+e^* b)+e^*(\kappa_A c+|A|-\beta_2). \end{aligned}$$

We obtain

$$L(A)+L^*(A)=\kappa_A\omega+\kappa_A^*\omega^*+|A|(e+e^*)-(e\beta_1+e^*\beta_2),$$

where $\omega = \alpha_2 - \alpha_1 - a - 1 + r + ea + e^*c$ and $\omega^* = \gamma_2 - \gamma_1 - b - 1 + r^* + e^*b + ec$.

On other hand

$$L(A) + L^*(A) = \sum_{x \in X} \sum_{y \in A} \lambda(x, y),$$

and we will evaluate this sum by counting in two ways triple (x, y, B) with $B \in \mathcal{A}, x \in B$, and $y \in A \cap B$:

$$\sum_{x \in X} \sum_{y \in A} \lambda(x, y) = \sum_{B \in \mathcal{A}} |B \cap A| \cdot |B| = |A|^2 + \lambda \sum_{B \in \mathcal{A} - \{A\}} |B|.$$

Since $\sum_{B \in \mathcal{A}} |B| = er + e^*r^*$, we obtain that

$$|A|^2 + \lambda(er + e^*r^* - |A|) = \kappa_A \omega + \kappa_A^* \omega^* + |A|(e + e^*) - (e\beta_1 + e^*\beta_2). \tag{3.4}$$

We express κ_A and κ_A^* in terms of $|A|$ and transform the Equation (3.4) into a quadratic equation for $|A|$. This $|A|$ may have at most two distinct values and, since D is λ-design, it has exactly two distinct values. □

In view of the 2^{nd} part of Theorem 2.1 we prove the following theorem.

Theorem 3.3. *Let $D = (X, \mathcal{A})$ be a λ-design of index λ with replication numbers r and r^*. If $\gcd(r - r^*, r + r^*) = s$, where s is positive integer and $\rho \neq \frac{2s+k}{k}$,$1 \leq k \leq s - 1$, then, D is of type-1.*

Proof. Let $c = (r-1)/g, c^* = (r^*-1)/g$. As $(r - r^*) = g(c - c^*), 4\lambda = v + 1 = r + r^*$, we get $m = \gcd(c - c^*, \lambda)$ divides s. Hence by the Theorem 3.1 if $\rho \neq \frac{2m'+k}{k}$, where $1 \leq k \leq m' - 1$ and m' divides s, then D is of type-1. To complete the proof we observe that $\left\{\frac{2m'+k}{k} \mid m' \text{ divides } s, 1 \leq k \leq m' - 1\right\} = \left\{\frac{2s+k}{k} \mid 1 \leq k \leq s - 1\right\}$. □

Corollary 3.1. *If $r \leq 3r^* + 6$, then D is of type-1.*

Proof. Observe that $\rho \leq 3$ if and only if $r \leq 3r^* - 2$. If $r = 3r^* - 1$, then $s = \gcd(r - r^*, r + r^*) = 1$. If $r = 3r^* + \tau$, where $\tau = 0, 1, \ldots, 6$, then $g = \gcd(r - 1, r^* - 1) \leq 8$. Proof follows from Theorems 3.1, and 2.1. □

Corollary 3.2. *If $v \leq 4r^* + 5$, then D is of type-1.*

Proof. Proof follows from the relation $v = r + r^* - 1$ and Corollary 3.1. □

Remark 3.3. *Let D be a λ-design of index λ with replication numbers r and $r^* (r > r^*)$, with $g = \gcd(r - 1, r^* - 1)$. If $r = \tau_1 r^* + \tau_2$, where τ_1 and τ_2 are positive integers then observe that g divides $\tau_1 + \tau_2 - 1$. Hence by Theorem 2.1, if $0 \leq \tau_1 + \tau_2 - 1 \leq 8$, then D is of type-1. If $\tau_2 = 1$, then $\gcd(r - r^*, r + r^*) = 1$ or 2, hence D is of type-1.*

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Corollary 3.4. *If $\lambda \leq 3$, then D is of type-1.*

Proof. Proof follows from the part (1) of Theorem 3.1 and inequality (2.1). □

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Ajeet Kumar Yadav

Department of Mathematics, St. Gonsalo Garcia College, University of Mumbai, India

Email: yadav.ajeet92@gmail.com

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