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EQUABLE KITES, TRAPEZOIDS AND CYCLIC QUADRILATERALS ON THE EISENSTEIN LATTICE

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Abstract. We show that on the Eisenstein lattice, up to Euclidean motions, there is only one infinite family of equable kites, which is given by the Pell-like equation $3x^2 - 2 = y^2$, and only one single equable trapezoid, which also happens to be the only equable cyclic quadrilateral.

1. Introduction

A polygon is *equable* if its area and perimeter are equal. In this paper we study equable quadrilaterals whose vertices lie on the Eisenstein lattice $\mathbb{Z}[\omega]$. Recall that $\mathbb{Z}[\omega] = \{x + y\omega : x, y \in \mathbb{Z}\} \subseteq \mathbb{C}$, where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We restrict ourselves here to the study of equable kites, trapezoids, and cyclic quadrilaterals on the Eisenstein lattice.

This paper is a continuation of a program of work which commenced in [3] with the treatment of equable parallelograms on the Eisenstein lattice. The overall goal is to compare the behavior of equable polygons on the Eisenstein lattice with those on the integer lattice. The results and proofs of the present paper are very similar in nature to those of [2], which treated equable kites, trapezoids, and cyclic quadrilaterals with vertices on the integer lattice. The main conclusion of the present paper is that the situation on the Eisenstein lattice is more restrictive than that on the integer lattice. On the

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integer lattice, we saw in [2] that, up to Euclidean motions, there are 4 infinite families of equable kites, exactly 5 equable trapezoids (2 isosceles, 2 right, 1 singular) and 4 equable cyclic quadrilaterals. On the Eisenstein lattice, this present paper will show that there is only one infinite family of equable kites, and only one single equable trapezoid, which also happens to be the only equable cyclic quadrilateral. This continues a theme that we have seen in our previous work. There are only 2 equable triangles on the Eisenstein lattice [4], while there are 5 equable triangles on the integer lattice; see [6, pp.15–16]. There is only one infinite family of equable parallelograms on the Eisenstein lattice [3], while there are 3 infinite families of equable parallelograms on the integer lattice [1].

We now present our results. Recall that a *kite* is a planar quadrilateral having two disjoint pairs of adjacent sides of equal length. A *trapezoid* is a quadrilateral that has exactly one pair of parallel sides. An *isosceles trapezoid* is a trapezoid for which the nonparallel sides have equal length. A *cyclic quadrilateral* is a quadrilateral whose vertices lie on a circle.

Theorem 1.1. *Up to Euclidean motions, the only equable kites on the Eisenstein lattices are those in the infinite family with vertices as follows:*

$$O,\ A = 3n + 3t - 2 - 4\omega,\ B = 6t,\ C = 3n + 3t + 2 + 4\omega,$$

where $n, t \in \mathbb{N}$ *and* $3t^2 - 2 = n^2$.

The first two solutions of $3t^2 - 2 = n^2$ are $(n, t) = (1, 1)$ and $(n, t) = (5, 3)$. The corresponding kites of Theorem 1.1 are shown in Figure 1, where the first one is reflected in the y-axis. Note that the first one is degenerate in that two successive sides are collinear; this kite is obtained by introducing a vertex in the midpoint of one of the sides of the unique equable equilateral triangle on the Eisenstein lattice (see [4]).

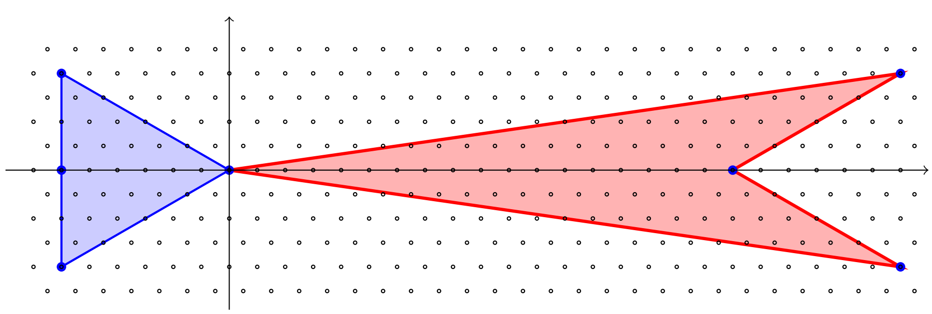

FIGURE 1. Two equable kites on the Eisenstein lattice

Corollary 1.2. *Up to Euclidean motions, there is only one convex lattice equable kite, namely the kite with* $(n, t) = (1, 1)$ *in Theorem 1.1.*

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Theorem 1.3. *Up to Euclidean motion, there is only one equable trapezoid on the Eisenstein lattice. It is the isosceles trapezoid shown in Figure 2, with vertices $O, 4+2\omega, 7+8\omega, 10+10\omega$. It has perimeter and area equal to $12\sqrt{3}$.*

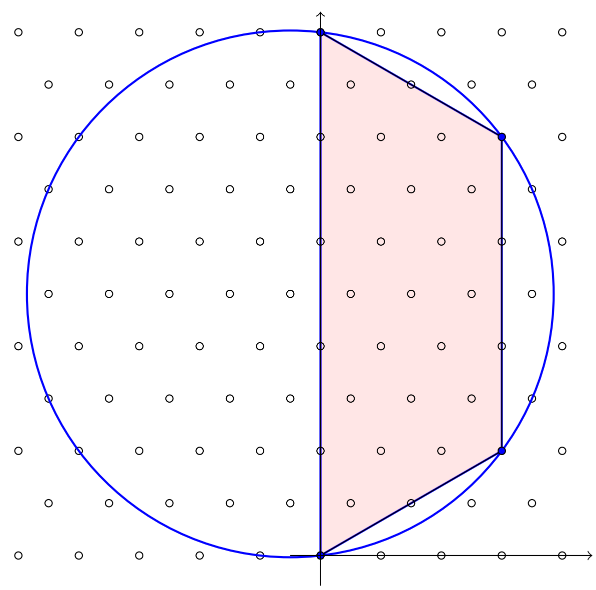

FIGURE 2. Equable isosceles trapezoid on the Eisenstein lattice

Theorem 1.4. *Up to Euclidean motions, there is only one cyclic equable quadrilateral on the Eisenstein lattice. It is the isosceles trapezoid of Theorem 1.3.*

The proofs of Theorem 1.1, 1.3 and 1.4 are given in Sections 2, 3 and 4 respectively. The following lemma provides a key fact about equable polygons on the Eisenstein lattice we will use repeatedly in this paper.

Lemma 1.5. [3, Lemma 1] *If P is an equable polygon with vertices in $\mathbb{Z}[\omega]$, then the side lengths of P are each of the form $n\sqrt{3}$, for some $n \in \mathbb{N}$.*

Observe that if P is an equable polygon, then by the above lemma, the area of P is also in $\sqrt{3}\mathbb{N}$.

Notation 1.6. *For a quadrilateral $OABC$, the side lengths of OA, AB, BC, CO are denoted $a\sqrt{3}, b\sqrt{3}, c\sqrt{3}, d\sqrt{3}$, respectively, where $a, b, c, d \in \mathbb{N}$. The lengths of the diagonals OB, AC are denoted p, q, respectively. For the various polygons in this paper, we use the symbol K for area and P for perimeter. For an Eisenstein lattice point $x+y\omega$, we find it is more convenient in this work to write it equivalently as the point $(x - \frac{y}{2}, \frac{y}{2}) \in \mathbb{R}^2$. We only use the notation $x + y\omega$ when we want to make explicit the fact that the point lies in $\mathbb{Z}[\omega]$. In this paper, we employ the term* positive *in the strict sense. So $\mathbb{N} = \{n \in \mathbb{Z} \mid n > 0\}$.*

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2. Proof of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Consider an equable kite on the Eisenstein lattice with vertices

$$O(0,0), A(x,y), B(z,w), C(u,v),$$

in positive cyclic order, such that the kite is symmetrical about the diagonal $p = OB$. By translation, and reflection in the x-axis and y-axis, if necessary, we may assume without loss of generality that the diagonal p lies in the (closed) first quadrant and that $a \geq b$.

Notice that $O(0,0), A(x,y), B(z,w), C'(z-x, w-y)$ is an equable parallelogram on the Eisenstein lattice. We will employ the results of [3]. From [3, Lemma 2],

$$p^2 = 3(a^2+b^2) \pm 2\sqrt{9a^2b^2 - 12(a+b)^2}, \tag{2.1}$$

and from [3, Lemma 6b], the diagonal $AC = q$ satisfies

$$q^2 = \frac{48(a+b)^2}{p^2}. \tag{2.2}$$

Furthermore, from [3, Theorem 2], there are coprime positive integers s, t for which

$$ab = 2(s^2+3t^2), \quad a+b = 6st. \tag{2.3}$$

Note that

$$\sqrt{9a^2b^2 - 12(a+b)^2} = \sqrt{36(s^2+3t^2)^2 - 12\cdot 36s^2t^2} = 6|s^2-3t^2|. \tag{2.4}$$

Hence, from (2.1), (2.3) and (2.4),

$$p^2 = 3\cdot 36s^2t^2 - 12(s^2+3t^2) \pm 12(s^2-3t^2). \tag{2.5}$$

We need to consider the two possibilities for the $\pm$ sign in (2.5).

First suppose that

$$p^2 = 3\cdot 36s^2t^2 - 12(s^2+3t^2) + 12(s^2-3t^2) = 3\cdot 36s^2t^2 - 2\cdot 36t^2. \tag{2.6}$$

Then from (2.2),

$$q^2 = \frac{48(a+b)^2}{p^2} = \frac{48\cdot 36s^2t^2}{3\cdot 36s^2t^2 - 2\cdot 36t^2} = \frac{48s^2}{3s^2-2} = 16 + \frac{32}{3s^2-2}. \tag{2.7}$$

Since q^2 is the square of the distance between two Eisenstein lattice points, it is a positive integer, though q may well be irrational. From (2.7), as q^2 is an integer, $3s^2-2$ is a factor of 32. As s is an integer, it follows that $s = 1$. Then (2.7) gives $q^2 = 48$, so $q = 4\sqrt{3}$. In this case, one has $a = 3t + \sqrt{3t^2-2}, b = 3t - \sqrt{3t^2-2}$, from (2.3). Thus, we require that $3t^2-2$ be a square, say $3t^2 - 2 = n^2$. So $a = n + 3t, b = -n + 3t$. From (2.6), $p^2 = 36t^2$, so $p = 6t$. Note that these values for a, b, p, q are realized by the vertices $O, A = (3n+3t, -2\sqrt{3}), B = (6t, 0), C = (3n+3t, 2\sqrt{3})$, as

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claimed in the statement of Theorem 1.1. Indeed, for these points one obviously has $p = 6t, q = 4\sqrt{3}$, and

$$3a^2 = OA^2 = (3n+3t)^2 + 12 = (3n+3t)^2 + 6(3t^2 - n^2) = 3(n+3t)^2,$$
$$3b^2 = AB^2 = (3n-3t)^2 + 12 = (3n-3t)^2 + 6(3t^2 - n^2) = 3(-n+3t)^2,$$

as required. To check that $OABC$ is equable, note that its perimeter is $2\sqrt{3}(a+b) = 12\sqrt{3}t$, while its area is $\frac{1}{2}pq = \frac{1}{2}\cdot 6t \cdot 4\sqrt{3} = 12\sqrt{3}t$.

Finally, it remains to eliminate the possibility that

$$p^2 = 3\cdot 36s^2t^2 - 12(s^2+3t^2) - 12(s^2-3t^2) = 3\cdot 36s^2t^2 - 24s^2.$$

Then from (2.2),

$$q^2 = \frac{48(a+b)^2}{p^2} = \frac{48\cdot 36s^2t^2}{3\cdot 36s^2t^2 - 24s^2} = \frac{48\cdot 3t^2}{9t^2-2} = 16 + \frac{32}{9t^2-2}. \tag{2.8}$$

From (2.8), as q^2 is an integer, $9t^2-2$ is a factor of 32. But this is impossible, as t is a positive integer. This completes the proof of Theorem 1.1. □

Remark 2.1. *The equation $3t^2-2 = n^2$ that appears in Theorem 1.1 is a well known Pell-like equation; see OEIS entry A001835 [8]. Its solutions (n_i, t_i) are given by the following recurrence relation:*

$$n_i = 4n_{i-1} - n_{i-2}, \quad \text{with } n_1 = 1, n_2 = 3,$$
$$t_i = 4t_{i-1} - t_{i-2}, \quad \text{with } t_1 = 1, t_2 = 5.$$

The first 10 kites of Theorem 1.1 are shown in Table 1.

Proof of Corollary 1.2. It suffices to note that for the kites of Theorem 1.1, the midpoint of the diagonal AC occurs at $(3n+3t, 0)$, which is to the right of $B = (6t, 0)$ for $t > 1$, because

$$3n+3t > 6t \iff n > t \iff n^2 > t^2 \iff 3t^2 - 2 > t^2 \iff t > 1,$$

for all $n, t \in \mathbb{N}$ with $n^2 = 3t^2 - 2$. □

3. Proof of Theorem 1.3

Consider an equable trapezoid $OABC$ with vertices on the Eisenstein lattice, in counterclockwise order, such that OA is the longest side, having an acute angle at 0; see Figure 3.

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TABLE 1. Examples of equable kites on the Eisenstein lattice

n	t	a	b	A	B
1	1	4	2	$4-4\omega$	6
5	3	14	4	$22-4\omega$	18
19	11	52	14	$88-4\omega$	66
71	41	194	52	$334-4\omega$	246
265	153	724	194	$1252-4\omega$	918
989	571	2702	724	$4678-4\omega$	3426
3691	2131	10084	2702	$17464-4\omega$	12786
13775	7953	37634	10084	$65182-4\omega$	47718
51409	29681	140452	37634	$243268-4\omega$	178086
191861	110771	524174	140452	$907894-4\omega$	664626

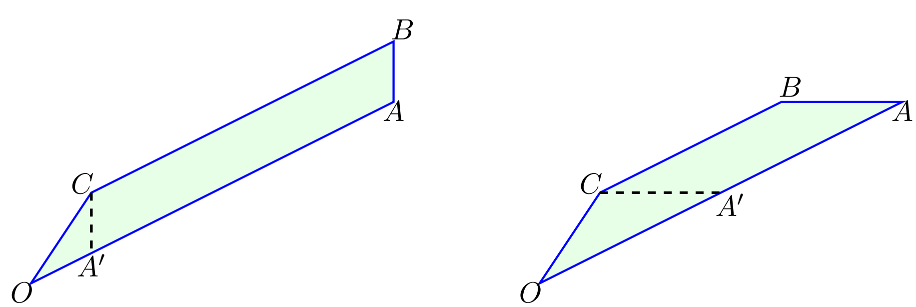

FIGURE 3. Trapezoids; obtuse and acute

Let A' be the point of the side OA such that $A'C$ is parallel to AB. Let $a\sqrt{3}, c\sqrt{3}$ be the lengths of sides OA, BC respectively, and let h be the distance between the parallel side OA, BC. Note that the area $K(OABC)$ is $h(a+c)\sqrt{3}/2$, and so h is rational, since $K(OABC)$ is in $\sqrt{3}\mathbb{N}$. The triangle $OA'C$ has sides of length in $\sqrt{3}\mathbb{N}$. Its area $K(OA'C)$ is $h(a-c)\sqrt{3}/2$, which is in $\sqrt{3}\mathbb{Q}$. To see that $K(OA'C)$ is in $\frac{\sqrt{3}}{4}\mathbb{N}$, we employ the following lemma.

Lemma 3.1. *If a triangle has sides of length in $\sqrt{3}\mathbb{N}$ and its area is in $\sqrt{3}\mathbb{Q}$, then its area is in $\frac{\sqrt{3}}{4}\mathbb{N}$.*

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Proof. Consider a triangle with integer sides $x\sqrt{3}, y\sqrt{3}, z\sqrt{3}$ and area $T\sqrt{3}$, where T is rational. By Heron's formula [7, Chap. 6.7],

$$(4T)^2 = 3(x+y+z)(-x+y+z)(x-y+z)(x+y-z).$$

Since $3(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$ is an integer, and its square root, $4T$, is rational, $4T$ is necessarily an integer. □

Since $OABC$ is equable, its area is greater than the sum of the lengths of the parallel sides; that is, $h(a+c)\sqrt{3}/2 > (a+c)\sqrt{3}$. So $h > 2$. Now the area $K(OABC)$ is $K(OA'C) + hc\sqrt{3}$ and the perimeter $P(OABC)$ is $P(OA'C) + 2c\sqrt{3}$. Hence, as $OABC$ is equable,

$$P(OA'C) = K(OA'C) + (h-2)c\sqrt{3} > K(OA'C).$$

In other words, $OA'C$ is perimeter-dominant. Thus $OA'C$ satisfies the hypotheses of the main theorem of [5]. Consequently, $OA'C$ is one of the triangles given in Table 2. That is, $OA'C$ is either the equilateral triangle of side length $3\sqrt{3}$, or a member of one of the three infinite families of triangles in cases a,b,c of Table 2.

TABLE 2. Perimeters and areas

Case	Relation	Side Lengths/$\sqrt{3}$	Perimeter/$\sqrt{3}$	Area/$\sqrt{3}$
		$(3,3,3)$	9	$27/4$
a	$1 = 4x^2 - 3y^2$	$(x,x,1)$	$2x+1$	$3y/4$
b	$1 = x^2 - 3y^2$	$(x,x,2)$	$2x+2$	$3y$
c	$1 = 4x^2 - 15y^2$	$(3x+1, 3x-1, 3)$	$6x+3$	$45y/4$

For an equable trapezoid $OABC$, as above, let $f = a - c$, so that $f\sqrt{3}$ is the length OA'. Then from above, $P(OA'C) = K(OA'C) + (h-2)c\sqrt{3} = K(OA'C) + (\frac{2K(OA'C)}{f\sqrt{3}} - 2)c\sqrt{3}$. For ease of notation, let us write $P(OA'C) = \rho\sqrt{3}$ and $K(OA'C) = \alpha\sqrt{3}$, where ρ and 4α are integers. Rearranging the above equation, we have

$$c = \frac{f\,(\rho - \alpha)}{2(\alpha - f)}. \tag{3.1}$$

The constraint that eliminates triangles is the requirement that c is an integer.

For the equilateral triangle of side length $3\sqrt{3}$, we have $f = 3, \rho = 9$ and $\alpha = \frac{27}{4}$. Then (3.1) gives $c = \frac{9}{10}$, contradicting the assumption that c is a positive integer. So no equable trapezoid on the Eisenstein lattice is obtained in this case.

It remains for us to consider each of the cases a,b,c of Table 2.

Case a: Here $1 = 4x^2 - 3y^2$. We have $\rho = 2x + 1$ and $\alpha = 3y/4$. So by (3.1),

$$c = \frac{f(-4 - 8x + 3y)}{8f - 6y}.$$

- If $f=1$, then $c=\frac{4+8x-3y}{6y-8}=\frac{4x}{3y-4}-\frac{1}{2}$. As $1=4x^2-3y^2$, y and $3y-4$ are both odd. But one easily sees that the difference of two rationals of opposite parity for both numerator and denominator is not an integer.
- If $f=x$, then $c=\frac{x(-4-8x+3y)}{8x-6y}$. As $1=4x^2-3y^2$, we have $y<2x/\sqrt{3}$. So $8x-6y>x(8-12/\sqrt{3})>x$ implies $c<-4-8x+3y<-4-8x+2\sqrt{3}x<-4-4x<0$, a contradiction. Hence, there is no equable trapezoid on the Eisenstein lattice in case a.

Case b: Here $1=x^2-3y^2$. We have $\rho=2x+2$ and $\alpha=3y$. So by (3.1),

$$c=\frac{f(-2-2x+3y)}{2(f-3y)}.$$

- If $f=2$, then

$$c=\frac{-2-2x+3y}{2-3y}=\frac{2x}{3y-2}-1=\frac{2\sqrt{3y^2+1}}{3y-2}-1.$$

Calculations show that for $y>1$ one has $c<1$, contrary to our hypothesis that $c\in\mathbb{N}$. For $y=1$, the equation $1=x^2-3y^2$ has the solution $x=2$. This gives $c=3$. As $x=2$, the triangle $OA'C$ is equilateral. We have $a=c+x=5$, and h, which is the altitude of $OA'C$ is 3. The resulting trapezoid is shown in Figure 2; it has vertices $O,(3,\sqrt{3}),(3,4\sqrt{3}),(0,5\sqrt{3})$. It is isosceles, and hence cyclic. The circumcircle has centre $(\frac{-1}{2},\frac{5\sqrt{3}}{2})$ and radius $\sqrt{19}$.
- If $f=x$, then $c=\frac{x(2+2x-3y)}{2(3y-x)}$. For $x=1$, the equation $1=x^2-3y^2$ gives $y=0$, but we have assumed that $y>0$. For $x=2$, we have $y=1$ and $c=3$. This is the same solution we found in the $f=2$ case above. So we may assume that $x\geq 3$. Then $y^2=\frac{1}{3}x^2-1\geq\frac{8}{3}$. So $y\geq 2$. But $1=x^2-3y^2$ has no integer solution for $y=2$. So we assume that $y\geq 3$. As $1=x^2-3y^2$, we have $\gcd(x,3y-x)=1$. So $3y-x$ divides $2+2x-3y$ and hence it also divides $x+2$. Let $c'=\frac{x+2}{3y-x}$. As we have just seen, $c'\in\mathbb{N}$. Now $1=x^2-3y^2$ gives $1=(x-\sqrt{3}y)(x+\sqrt{3}y)$, so $x-\sqrt{3}y<1$. Thus $\sqrt{3}y>x-1$, so $3y-x>(\sqrt{3}-1)x-\sqrt{3}$, which is positive for $x\geq 3$. Thus, for integers $x,y\geq 3$,

$$\begin{aligned} c'=\frac{x+2}{3y-x}>1 &\iff 2x>3y-2 \iff 4x^2>(3y-2)^2\\ &\iff 4+12y^2>(3y-2)^2 \quad (\text{as } 1=x^2-3y^2)\\ &\iff 3y^2>-12y, \end{aligned}$$

which holds for all $y\in\mathbb{N}$. But

$$\begin{aligned} c'=\frac{x+2}{3y-x}<2 &\iff 3x<6y-2\\ &\iff 9(1+3y^2)<(6y-2)^2 \iff 0<9y^2-24y-5, \end{aligned}$$

which holds for all $y\geq 3$. Thus $1<c'<2$, which is impossible for $c'\in\mathbb{N}$.

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Case c: Here $1 = 4x^2 - 15y^2$. Note that y is necessarily odd. We have $\rho = 6x + 3$ and $\alpha = 45y/4$. So by (3.1),

$$c = \frac{-3f(4 + 8x - 15y)}{8f - 90y}.$$

- If $f = 3$, then $c = \frac{3(4+8x-15y)}{30y-8}$. Notice that as c is an integer and $30y - 8$ is even, $4 + 8x - 15y$ must also be even. But this is impossible since y is odd.
- We perform the last two cases together. If $f = 3x \pm 1$, then $c = \frac{3(3x\pm1)(4+8x-15y)}{90y-24x\mp8}$. As c is an integer, the divisors of the denominator are also divisors of the numerator. First note that as y is odd, $4 + 8x - 15y$ is odd. So, as $90y - 24x \mp 8$ is even, $3x \pm 1$ must be even. Hence x must be odd. Let $x = 2X + 1, y = 2Y + 1$. Then $1 = 4x^2 - 15y^2$ gives $12 = 16X + 16X^2 - 60Y - 60Y^2$, so $3 = 4X + 4X^2 - 15Y - 15Y^2$. But modulo 2, this gives $1 \equiv Y + Y^2$, which is impossible.

This completes the proof of Theorem 1.3.

4. Cyclic quadrilaterals; Proof of Theorem 1.4

Consider an equable cyclic quadrilateral with (not necessarily successive) side lengths $a\sqrt{3}, b\sqrt{3}, c\sqrt{3}, d\sqrt{3}$, where $a \geq b \geq c \geq d$. By Brahmagupta's formula [7, Chap. 6.8],

$$3(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d) = 16(a + b + c + d)^2. \tag{4.1}$$

Notice each sum above has the same parity and hence is even. Let $2w = -a + b + c + d, 2x = a-b+c+d, 2y = a+b-c+d, 2z = a+b+c-d$ and note by elementary geometry $0 < w \leq x \leq y \leq z$. Hence

$$3wxyz = (w + x + y + z)^2. \tag{4.2}$$

Conversely,

$$\begin{aligned} a &= \frac{1}{2}(-w + x + y + z), \quad & b &= \frac{1}{2}(w - x + y + z), \\ c &= \frac{1}{2}(w + x - y + z), \quad & d &= \frac{1}{2}(w + x + y - z). \end{aligned} \tag{4.3}$$

Since $w \leq x \leq y \leq z$, one has $w + x + y - z = 2d > 0$, and so

$$z < w + x + y. \tag{4.4}$$

From (4.2),

$$z = \frac{3wxy - 2(w + x + y) \pm \sqrt{9w^2x^2y^2 - 12wxy(w + x + y)}}{2}.$$

In particular, $9w^2x^2y^2 - 12wxy(w + x + y) \geq 0$, so

$$3wxy \geq 4(w + x + y). \tag{4.5}$$

Suppose for the moment that z is given by the positive square root. Then (4.4) would give

$$z = \frac{3wxy - 2(w + x + y) + \sqrt{9w^2x^2y^2 - 12wxy(w + x + y)}}{2} < w + x + y,$$

so $\sqrt{9w^2x^2y^2 - 12wxy(w+x+y)} < 4(w+x+y) - 3wxy \leq 0$, by (4.5). But this is impossible. So we may assume that

$$z = \frac{3wxy - 2(w+x+y) - \sqrt{9w^2x^2y^2 - 12wxy(w+x+y)}}{2}. \tag{4.6}$$

Then $y \leq z$ gives $\sqrt{9w^2x^2y^2 - 12wxy(w+x+y)} \leq 3wxy - 2(w+x+2y)$. Squaring and simplifying gives

$$3wxy^2 \leq (w+x+2y)^2. \tag{4.7}$$

Consequently, as $w \leq x \leq y$, we have $3wxy^2 \leq (4y)^2$, so $3wx \leq 16$. Thus $wx \leq 5$. Hence $w^2 \leq wx \leq 5$ and so $w \leq 2$. And $x \leq wx \leq 5$. Using (4.5) we have $3wx > 4$ and so $wx \geq 2$, and then rewriting (4.7) we get

$$y \leq \frac{w+x}{\sqrt{3wx}-2} \leq \frac{2+5}{\sqrt{6}-2} < 16.$$

There are 64 triples (w,x,y) satisfying $w \leq x \leq y \leq 15$ with $2 \leq wx \leq 5$. Testing each possibility by replacing it in (4.6), and checking if z is an integer and $y \leq z < w+x+y$, we obtain only one solution for (w,x,y,z), namely

$$(w,x,y,z) = (1,3,4,4),$$

which by (4.3), give the corresponding solution announced for (a,b,c,d):

$$(a,b,c,d) = (5,3,2,2). \tag{4.8}$$

This is realized by the isosceles trapezoid with complex vertices $O, 3+\sqrt{3}i, 5\sqrt{3}i, 3+4\sqrt{3}i$; see Figure 2. It has sides in the order $2\sqrt{3}, 3\sqrt{3}, 2\sqrt{3}, 5\sqrt{3}$. It is not difficult to see that there is no cyclic quadrilateral on the Eisenstein lattice with side lengths in the order $2\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, 5\sqrt{3}$. This completes the proof of Theorem 1.4.

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