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SOME DESIGNS FROM THE FIXED POINTS OF ALTERNATING GROUPS

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Abstract. In this paper, we construct some $1-(v,k,\lambda)$ designs from the alternating group $G = A_n$ with the maximal subgroup isomorphic to $M = A_{n-1}$. The method we use is called Key-Moori Method 2. Furthermore, from the set I_x which is the intersection of all blocks containing the point $x \in G$, we construct corresponding reduced designs. Our aim is to give explicit formulae to compute the parameters of the designs based on the cyclic structures of the permutations in G.

1. Introduction

Two methods for constructing designs and codes from finite simple groups were introduced in [2, 4] and summarized in [7]. The first method is about primitive permutation representation of finite simple groups in constructing symmetric 1−designs. The second method presents a technique from which a large number of non-symmetric 1−designs can be constructed on a maximal subgroups and conjugacy classes of elements of finite groups. Both methods where then applied to many finite simple groups,(see [1, 6, 9, 8, 11, 10]).

In [5] Le and Moori introduced the concepts of reduced designs from Method 2. If $\mathcal{D}$ is a $1-(v,k,\lambda)$ design constructed by Key-Moori Method 2, and I_x is the intersection of all blocks containing the point x of the design, then we have a $1-(\frac{v}{|I_x|}, \frac{k}{|I_x|}, \lambda)$ design $\mathcal{D}_I$ called the reduced design of $\mathcal{D}$. In [12],

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the second author used the set I_x for some special cases to find the automorphisms of each design constructed from $PSL_2(q)$ and $Sz(q)$, were q is a power of 2.

In section 3, we determine the parameters of the designs constructed by Key-Moori Method 2 and find the set I_x for the family of alternating groups from which the parameters of the designs are constructed. The main results of the $1-(v,k,\lambda)$ designs from Method 2 under the family of alternating groups with maximal subgroup isomorphic to A_{n-1} can be summarized in the following theorem.

Theorem 1.1. *Let $G = A_n$, and let M be its maximal subgroup isomorphic to A_{n-1}. If $g \in G$, then there exists $1-(v,k,\lambda)$ design $\mathcal{D}$, where $v = |g^G|$,*

$$k = \frac{\gamma(n-1)!}{\prod_{i=1}^{k} (u_i)^{r_i} \times r_i!}$$

and $\lambda = \gamma$ is the number of fixed points of g in G. Moreover, if $x \in g^G$, its associated reduced design is a $1-(\frac{v}{I_x}, \frac{k}{I_x}, \lambda)$ design. The group G, acts transitively on the point set of $\mathcal{D}$ and primitively on the block set of $\mathcal{D}$.

Remark 1.2. *Its not necessarily the case that all the design $\mathcal{D}$ constructed above are isomorphic as g runs through elements of G.*

Throughout the rest of this paper, we consider the ordinary action of the alternating group A_n on the set of n points. This is a primitive action whose point-stabilizer is a maximal subgroup isomorphic to A_{n-1}. Let Fix(g) be the set of fixed points of g with respect to ordinary action of A_n on n points.

2. Preliminaries

Our notation for designs and codes will be standard as in [3], and $\mathbb{ATLAS}$ [14] for finite simple groups and their maximal subgroups. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure with point set $\mathcal{P}$, block set $\mathcal{B}$ and $\mathcal{I}$ is a $t-(v,k,\lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The complement of $\mathcal{D}$ is the structure $\overline{\mathcal{D}} = (\overline{\mathcal{P}}, \overline{\mathcal{B}}, \overline{\mathcal{I}})$ where $\overline{\mathcal{P}} = p$, $\overline{\mathcal{B}} = b$ and $\overline{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The dual structure of $\mathcal{D}$ is $D^t = (\mathcal{B}^t, \mathcal{P}^t, \mathcal{I}^t)$ where $\mathcal{P}^t = \mathcal{B}$, $\mathcal{B}^t = \mathcal{P}$. A design is said to be **symmetric** if it has the same number of points and blocks and **self − dual** if it is isomorphic to its dual. The point-set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the blocks set by $\mathcal{B}$. The length of the code is the cardinality of $\mathcal{P}$ and its dimension is the rank of the incidence matrix of the design $\mathcal{D}$.

For every $g \in S_n$, we can write g as a product of disjoint cycles. Let $\mu(g)$ be the cycle structure of g, that is a multiset whose underlying set is all the lengths of the cycle of g.

Definition 2.1. (**Multiset**). *A multiset is a 2-tuple $\mathcal{M}(U, r)$, where U is the underlying set and r is the multiplicity function $r : U \to \mathbb{Z}^+$.*

From [13] we have the following well known results.

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Definition 2.2. *Let $\mathcal{M}(U,r)=\{u_1^{r_1},u_2^{r_2},\dots,u_k^{r_k}\}$ be a multiset and $u_1r_1+u_2r_2+\cdots+u_kr_k=m$. We define $\mathcal{S}(U,r)$ as follows:*

$$\mathcal{S}(U,r)=\frac{\left(\begin{array}{c} m \\ \mathcal{M}(U,r)\end{array}\right)\times\prod_{i=1}^{k}(u_i-1)!^{r^i}}{\prod_{j=1}^{k}r_j!}$$

.

Lemma 2.3. *Let $g\in S_n$ and $\mu(g)=\{u_1^{r_1},u_2^{r_2},\dots,u_k^{r_k}\}$. Then we have,*

$$|g^G|=\frac{n!}{\prod_{i=1}^{k}(u_i)^{r_i}\times r_i!}.$$

Note 1. *If $g\in A_n$, then either $|g^{A_n}|=|g^{S_n}|$ or $|g^{A_n}|=\frac{|g^{S_n}|}{2}$. The latter happens only if $r_i=1$ for all $1\leq i\leq k$ and u_i are odd.*

3. Main Results

In this section we give the proof of main proposition outlined in the introduction. The proof is divided into two main subsections below.

3.1. **Designs From Method** 2**.** In this subsection, we use Method 2 to obtain the designs from the maximal subgroup M isomorphic to A_{n-1} of $G=A_n$. This method is based on the following well-known results.

Proposition 3.1. *(**Key − Moori Method 2**). Let G be a finite simple group, and M be a maximal subgroup of G. Let g^G be a conjugacy class of G containing g such that $M\cap g^G\neq\emptyset$. If $\mathcal{P}=g^G$ and $\mathcal{B}=\{(M\cap g^G)^y|y\in G\}$, then the incidence structure $(\mathcal{P},\mathcal{B})$ is a $1-(|g^G|,|M\cap g^G|,\gamma)$ design admitting G as its point-transitive and block-primitive automorphism group.*

Proof. See [7, Theorem 22]. □

The following result shows that if we have two parameters, then the third one can always be computed.

Lemma 3.2. *Let $\mathcal{D}=1-(v,k,\lambda)$ be a design obtained by Method 2. Then $|G:M|=\lambda v/k$.*

Proof. See [10, Lemma 3.2]. □

Theorem 3.3. *Let $G=A_n$ and M be a point-stabilizer of G such that for any $g\in G$, let γ be the number of fixed points of g in G. Then the design constructed by Method 2 is a*

$$1-\left(\frac{n!}{\prod_{i=1}^{k}(u_i)^{r_i}\times r_i!}, \frac{\gamma(n-1)!}{\prod_{i=1}^{k}(u_i)^{r_i}\times r_i!}, \gamma\right)$$

design.

Proof. Suppose that $\mathcal{D}$ is a $1-(v,k,\lambda)$ design and $g \in G$ with $\mu(g)=\{u_1^{r_1}, u_2^{r_2}, \ldots, u_k^{r_k}\}$. Since the set of fixed points of the elements g^G constitute the point set of the design, the first parameter is the length of g^G, that is,

$$|g^G| = \frac{n!}{\prod_{i=1}^{k}(u_i)^{r_i}\times r_i!}.$$

Since G acts on the set of points of M by conjugation then in this case λ is the number of fixed points of g in G, that is $\lambda=\gamma$. To complete the proof, we only need to find the parameter $k=|g^G \cap A_{n-1}|$. By the results of Lemma 3.7 we have

$$|G:M| = \frac{\lambda v}{k} n = \frac{\gamma |g^G|}{k}$$

Therefore,

$$k = \frac{\gamma \times n!}{n \times \prod_{i=1}^{k}(u_i)^{r_i}\times r_i!} = \frac{\gamma(n-1)!}{\prod_{i=1}^{k}(u_i)^{r_i}\times r_i!}.$$

This completes the proof. □

3.2. **Reduced Designs From Method** 2**.** Following the results of Method 2 above, we construct the corresponding reduced designs. We first need the following definition.

Definition 3.4. *Let $\mathcal{D}$ be a $1-(v,k,\lambda)$ design and let x be a point of the design contained in the blocks $B_1, B_2, \ldots, B_\lambda$. We define,*

$$I_x = \bigcap_{i=1}^{\lambda} B_i,$$

where x is the representative of I_x, that is, $I_x = I_y$ for all $y \in I_x$.

Definition 3.5. *Let $G \leq S_n$ be a finite transitive permutation group. For some $g \in G$ let $x \in g^G$, then $S(x) = \{g \in G : Fix(g) = Fix(x)\}$.*

Lemma 3.6. *Let G be a transitive group and $x \in G$. Then we have $|S(x)| = |I_x|$.*

Proof. See [12, Lemma 2.11]. □

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Theorem 3.7. *Let $\mathcal{D}$ be a $1-(v,k,\lambda)$ design constructed by applying Method 2 to the primitive action of $G = A_n$ on the n points. For $g \in G$, let $|Fix(g)| = \gamma$. Then we have*

$$|I_x| = \frac{(n-\gamma)!}{\prod_{i=2}^{n}(u_i^{r_i}) \times r_i!},$$

and with the case where the cycle split in A_n we have the following results

$$|I_x| = \frac{(n-\gamma)!}{2\prod_{i=2}^{n}(u_i^{r_i}) \times r_i!}.$$

Proof. Since by Lemma 3.6 we have $S(x) = I_x$, then we will prove I_x by finding $S(x)$. Let $g \in A_n$ and

$$\mu(g) = (1^{n-u_2r_2-u_3r_3\cdots u_nr_n}\ u_2^{r_2}\ u_3^{r_3}\cdots u_n^{r_n})$$

be the cycle structure of g with

$$|Fix(g)| = n - u_2r_2 - u_3r_3\cdots u_nr_n.$$

Let $\gamma = |Fix(g)|$ and g' be a fixed point free element of $H = A_{n-m}$, whose cyclic structure is the same as g. We claim that for a permutation h in A_n, we have $h \in S(x)$ if and only if h' is an element of g'^H. Suppose that $h \in S(x)$, then since $h \in A_n$ there is $h' \in A_{n-m}$ whose cyclic structure is same as that of h. Which would mean $|Fix(h)| = |Fix(h')|$, and thus h is a fixed point free element. To find all the possibilities for h, consider $\mu(h)$, the cycle structure of h with $u_1r_1 + u_2r_2 + \cdots + u_nr_n = n-m$. Then for every $h \in S(x)$, we should put aside $u_i = 1$, so that we can choose the u_i from $n-m$. Hence we can totally choose

$$\begin{pmatrix} n-\gamma \\ \mathcal{M}(U-\{1\},r) \end{pmatrix} = \frac{(n-\gamma)!}{\prod_{i=2}^{n}(u_i)^{r_i}}$$

elements for h. Since every cycle of length u_i have $(u_i-1)!$ permutations, now to avoid repetition we divide this by the number of the permutations of cycles with equal length $(r_i)!$. So if the conjugacy classes of h does not split, then the number of possible $h \in S(x)$ is equal to

$$S(U-\{1\},r) = \frac{(n-m)!}{\prod_{i=2}^{n}(u_i)^{r_i}(r_i)!}.$$

If the conjugacy classes of h split then,

$$S(U-\{i\},r)/2 = \frac{(n-\gamma)!}{2\prod_{i=2}^{n}(u_i)^{r_i}(r_i)!}.$$

But for every $h' \in H = A_{n-\gamma}$ we have

$$|g'^{A_{n-\gamma}}| = \frac{(n-\gamma)!}{\prod_{i=2}^{n}(u_i)^{r_i}(r_i)!}$$

by results of Lemma 2.3. Therefore, $|S(x)| = |g'^H|$ and the result follows by Lemma 2.3. □

Corollary 3.8. *Let $\mathcal{D}$ be a $1-(v,k,\lambda)$ design constructed by Method 2. Then we have the following corresponding reduced design*

$$\mathcal{D}_I = 1 - \left(\frac{n!}{(n-\gamma)!}, \frac{\gamma(n-1)!}{(n-\gamma)!}, \gamma\right).$$

Proof. The proof follows from the results of Theorems 3.3 and 3.7. □

4. some examples from main results

In this section we present some examples and computations from the results in section 3. For every design obtained, we give the corresponding complement design and also use MAGMA to obtain the corresponding codes.

Table 1. Some 1-designs from A_8

Conjugacy Classes of A_8	$\|Fix(g)\|$	$\|g^G\|$	$\frac{fv}{n}$	**1-design**	$Aut(\mathcal{D})$
(1 2)(3 4)	$\gamma = 4$	$v = 210$	$k = 105$	$1-(210,105,4)$	$2^{105}:S_8$
(1 2 3 4)(5 6)	$\gamma = 2$	$v = 2520$	$k = 630$	$1-(2520,630,2)$	$2^{1260}:S_8$
(1 2 3 4 5 6 7)	$\gamma = 1$	$v = 2880$	$k = 360$	$1-(2880,360,1)$	$(S_{360})^8:S_8$

4.1. Some examples of designs.

(1) The design $\mathcal{D} = 1-(210,105,4)$ is a self complementary design. Using MAGMA for construction, one can shows that the binary code of this design is $\mathcal{C}(\mathcal{D}) = [210,7,96]$ and the dual code of $\mathcal{C}$ is $[210,203]_2$.

(2) The complement of the design $\mathcal{D} = 1-(2520,630,2)$ is a $\tilde{\mathcal{D}} = 1-(2520,1890,6)$ design. Using MAGMA we get that $\mathcal{C}(\mathcal{D}) = [2520,7,630]_2$ and $\mathcal{C}(\tilde{\mathcal{D}}) = [2520,7,1080]_2$

(3) The complement of the design $\mathcal{D} = 1-(2880,360,1)$ is a $\tilde{\mathcal{D}} = 1-(2880,2520,7)$ design. From MAGMA we obtain that $\mathcal{C}(\mathcal{D}) = [2880,8,360]_2$ and $\mathcal{C}(\tilde{\mathcal{D}}) = [2880,8,360]_2$

In the following table, reduced designs $\mathcal{D}_I$ corresponding to Table 1 are given.

Table 2. Corresponding reduced 1-designs from Table 1

Conjugacy Classes of A_8	$\|Fix(g)\|$	$v = \frac{n!}{(n-f)!}$	$k = \frac{f(n-1)!}{(n-f)!}$	**1-design**	$Aut(\mathcal{D})$
(1 2)(3 4)	4	70	35	$1-(70,35,4)$	$(S_4 \times S_4):2$
(1 2 3 4)(5 6)	2	56	14	$1-(56,14,2)$	$S_5 \times S_3$
(1 2 3 4 5 6 7)	1	8	1	$1-(8,1,1)$	S_8

http://dx.doi.org/10.22108/toc.2025.141435.2178

Consider the $1-(|g^G|, |g^G \cap A_{n-1}|, \gamma)$ designs from our Method 2. Here we look at some results concerning the structure of the structure of $Aut(\mathcal{D})$ and $Aut(\mathcal{D}_I)$ together with their relationships. Le and Moori in [5] have already studied the automorphism of designs from the finite simple groups and deduced some general results. In relation they classified all the automorphism of G that could be lifted to automorphism of $\mathcal{D}$ and which cannot done by Method 2. This lifted automorphism forms a subgroup of Aut$\mathcal{D}$, called the $AD(G)$, where

$$AD(G) = \{\overline{\phi} : \phi \in Aut(G) \text{ such that } \phi(g) \in g^G \text{ and } \phi(M) = M^x \text{ for some } x \in G\}.$$

We firstly outline the following important result in [5] Theorem 2.12.

Theorem 4.1. *If $Aut(\mathcal{D})$ acts transitively on the point set and the block set, then*

$$Aut(\mathcal{D}_I) \cong Aut(\mathcal{D})/S(x).$$

Lemma 4.2. *For $\phi \in Aut(G)$ then $\overline{\phi} \in Aut(\mathcal{D})$ if and only if $\phi(g) \in g^G$ and $\phi(M) = M^x$ for some $x \in G$.*

Remark 4.3. *If g^G is the unique conjugacy class of elements of order n in G and G has only one maximal subgroup isomorphic to M, then $Aut(G) \cong AD(G) \leq Aut(D)$*

From Table 2 we determine the $Aut(\mathcal{D})$ as well as the $Aut(\mathcal{D}_I)$ of the reduced designs.

TABLE 3. The set I_x from designs in Table 1

Conjugacy Classes of A_8	$\lvert I_x \rvert$	$\mathcal{D}_I$ **: 1-design**	$Aut(\mathcal{D}_I)$
(1 2)(3 4)	3	$1-(70, 35, 4)$	$(S_4 \times S_4) : 2 \ncong A_8$
(1 2 3 4)(5 6)	45	$1-(56, 14, 2)$	$S_5 \times S_3 \ncong A_8$
(1 2 3 4 5 6 7)	360	$1-(8, 1, 1)$	S_8

Note 2. *If $\lambda = 1$, then all blocks are pairwise disjoint. And our $Aut(D) \cong S_k \wr S_b = S_I : S_b$ where b is the number of blocks and I is the intersection of blocks containing element x of the design.*

We are actually interested in the designs with $\lambda \geq 2$. From our table above we have,

(1) From $1-(210, 105, 4)$ design $\mathcal{D}$. For every point $x \in g^G$ the intersection of the blocks containing x has size 3. Therefore $\mathcal{D}_I$ is $1-(70, 35, 4)$ and $S(x) = (S_3)^{35} \lhd Aut(\mathcal{D})$. The automorphism group of $\mathcal{D}$ contains $(S_3)^{35} : (G : 2)$. Thus $Aut(\mathcal{D}) \cong (S_3)^{35} : (S_4 \times S_4) : 2$ and

$$Aut(\mathcal{D}_I) \cong Aut(\mathcal{D})/S(x) \cong (S_4 \times S_4) : 2 \ncong A_8.$$

(2) From $1-(2520, 630, 2)$ design $\mathcal{D}$, every point $x \in g^G$ the intersection of the blocks containing x has size 45. Therefore, $\mathcal{D}_I$ is $1-(56, 14, 2)$ and $S(x) = (S_{45})^{56} \lhd Aut(\mathcal{D})$. The automorphism group of $\mathcal{D}$ contains $(S_{45})^{56} : (G : 2)$. Thus $Aut(\mathcal{D}) \cong (S_{45})^{56} : S_5 \times S_3$ and

$$Aut(\mathcal{D}_I) \cong Aut(\mathcal{D})/S(x) \cong S_5 \times S_3 \ncong A_8.$$

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