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MODULAR CHROMATIC NUMBER OF $C_m \square P_n$

N. PARAMAGURU AND R. SAMPATHKUMAR*

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Abstract. A modular k-coloring, $k \geq 2$, of a graph G is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ having the property that for every two adjacent vertices of G, the sums of the colors of their neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k-coloring is the modular chromatic number of G. Except for some special cases, modular chromatic number of $C_m \square P_n$ is determined.

1. **Introduction**

For a vertex v of a graph G, let $N_G(v)$, the *neighborhood of* v, denote the set of vertices adjacent to v in G. For a graph G, let $c : V(G) \to \mathbb{Z}_k$, $k \geq 2$, be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v) = \sum_{u \in N_G(v)} c(u)$ of a vertex v of G is the sum of the colors of the vertices in $N_G(v)$. The coloring c is called a *modular k-coloring* of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices in G. The *modular chromatic number* $mc(G)$ of G is the minimum k for which G has a modular k-coloring. This concept was introduced by Okamoto, Salehi and Zhang [2].

The *Cartesian product* $G \square H$ of two graphs G and H has $V(G \square H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \square H$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

Okamoto, Salehi and Zhang proved in [2] that: every nontrivial connected graph G has a modular k-coloring for some integer $k \geq 2$ and $mc(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G; for the cycle C_n of length n, $mc(C_n)$ is 2 if $n \equiv 0 \mod 4$ and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph G, $mc(G) = \chi(G)$;

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∗Corresponding author.

for the Cartesian product $G = K_r \square K_2$, $mc(G)$ is r if $r \equiv 2 \mod 4$ and it is $r+1$ otherwise; for the wheel $W_n = C_n \vee K_1$, $n \geq 3$, $mc(W_n) = \chi(W_n)$, where $\vee$ denotes the join of two graphs; for $n \geq 3$, $mc(C_n \vee K_2^c) = \chi(C_n \vee K_2^c)$, where G^c denotes the complement of G; and for $n \geq 2$, $mc(P_n \vee K_2) = \chi(P_n \vee K_2)$, where P_n denotes the path of length $n-1$; and in [3] that: for $m, n \geq 2$, $mc(P_m \square P_n) = 2$.

In this paper, except for some special cases, we compute $mc(C_m \square P_n)$.

2. Result

For the path P_ν on ν vertices, and the cycle C_ν on ν vertices, let $V(P_\nu) = V(C_\nu) = \{1, 2, \ldots, \nu\}$, $E(P_\nu) = \{\{i, i+1\} : i \in \{1, 2, \ldots, \nu-1\}\}$ and $E(C_\nu) = E(P_\nu) \cup \{\{\nu, 1\}\}$. For $m \geq 3$ and $n \geq 2$, $\chi(C_m \square P_n)$ is 2 if m is even and it is 3 if m is odd. In [2], Okamoto, Salehi and Zhang proved that: *For every positive integer r, $r \leq mc(K_r \square K_2) \leq r+1$, and $mc(K_r \square K_2) = r$ if and only if $r \equiv 2$ mod 4.* Consequently, we have $mc(C_3 \square P_2) = 4$.

Theorem. Let $m \geq 3$ and $n \geq 2$.

(i) $mc(C_3 \square P_2) = 4$.

(ii) If neither

$m = 3$ and $n \in \{2, 14, 26, 38, \ldots, 12r+2, \ldots\} \cup \{16, 28, 40, \ldots, 12r+4, \ldots\} \cup \{8, 20, 32, \ldots, 12r+8, \ldots\} \cup \{22, 34, \ldots, 12r+10, \ldots\}$,

nor

$m \equiv 2 \mod 4$ and $n \equiv 1 \mod 4$,

then

$mc(C_m \square P_n) = \chi(C_m \square P_n)$.

(iii) If $m \equiv 2 \mod 4$ and $n \equiv 1 \mod 4$, then $mc(C_m \square P_n) \leq 3$.

(iv) If $n \equiv 1 \mod 4$, then $mc(C_6 \square P_n) = 3$.

Proof. (ii): First we consider $n = 2$; i.e., $C_m \square P_2$. If m is even, then the result follows from Proposition 3.6 (*If G is a bipartite graph the degrees of whose vertices are of the same parity, then $mc(G \square K_2) = 2$.*) of [2]. By hypothesis, $m \neq 3$. Hence, assume that $m \geq 5$ is odd. Define $c : V(C_m \square P_2) \to \mathbb{Z}_3$ as follows: $c((i,1)) = 1$ if $i \in \{1, 3, 5, \ldots, m-4\}$; $c((m-1, 2)) = 2$; $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,1)) = 0$ if $i \in \{1, 3, 5, \ldots, m-2\}$;

$\sigma((m-3, 1)) = \sigma((m, 1)) = 1$;

$\sigma((i,1)) = 2$ if $i \in \{2, 4, 6, \ldots, m-5\}$;

$\sigma((m-1, 1)) = 2$;

$\sigma((i,2)) = 0$ if $i \in \{2, 4, 6, \ldots, m-1\}$;

$\sigma((i,2)) = 1$ if $i \in (\{1, 3, 5, \ldots, m-4\}$;

$\sigma((m-2, 2)) = \sigma((m, 2)) = 2$.

Hence, c is a modular 3-coloring of $C_m \square P_2$.

This completes the proof for $n = 2$.

Next, we consider $m = 3$; i.e., $C_3 \Box P_n$. We prove the result by considering the following 4 cases.

Case 1. $n \equiv 1 \mod 4$.

Define $c : V(C_3 \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1\} \times \{3,7,11,\ldots,n-2\}) \cup (\{2\} \times \{1,5,9,\ldots,n\})$; $c((i,j)) = 2$ if $(i,j) \in (\{2\} \times \{3,7,11,\ldots,n-2\}) \cup (\{3\} \times \{1,5,9,\ldots,n\})$; and $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,j)) = 0$ if $(i,j) \in (\{1\} \times \{1,5,9,\ldots,n\}) \cup (\{2\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3\} \times \{3,7,11,\ldots,n-2\})$; $\sigma((i,j)) = 1$ if $(i,j) \in (\{1\} \times \{2,4,6,\ldots,n-1\}) \cup (\{2\} \times \{3,7,11,\ldots,n-2\}) \cup (\{3\} \times \{1,5,9,\ldots,n\})$; $\sigma((i,j)) = 2$ if $(i,j) \in (\{1\} \times \{3,7,11,\ldots,n-2\}) \cup (\{2\} \times \{1,5,9,\ldots,n\}) \cup (\{3\} \times \{2,4,6,\ldots,n-1\})$. Hence, c is a modular 3-coloring of $C_3 \Box P_n$. See Table 1 for colors of the vertices and color sums of the vertices for $C_3 \Box P_n$.

color c			color sum σ		
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 2 1	0 1 2 1...0 1 2 1	0
1 0 2 0	1 0 2 0...1 0 2 0	1	2 0 1 0	2 0 1 0...2 0 1 0	2
2 0 0 0	2 0 0 0...2 0 0 0	2	1 2 0 2	1 2 0 2...1 2 0 2	1

Table 1. $C_3 \Box P_n$ ($n \equiv 1 \mod 4$).

Case 2. $n \equiv 3 \mod 4$.

Define $c : V(C_3 \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1\} \times \{3,7,11,\ldots,n\}) \cup (\{2\} \times \{1,5,9,\ldots,n-2\})$; $c((i,j)) = 2$ if $(i,j) \in (\{2\} \times \{3,7,11,\ldots,n\}) \cup (\{3\} \times \{1,5,9,\ldots,n-2\})$; and $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,j)) = 0$ if $(i,j) \in (\{1\} \times \{1,5,9,\ldots,n-2\}) \cup (\{2\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3\} \times \{3,7,11,\ldots,n\})$; $\sigma((i,j)) = 1$ if $(i,j) \in (\{1\} \times \{2,4,6,\ldots,n-1\}) \cup (\{2\} \times \{3,7,11,\ldots,n\}) \cup (\{3\} \times \{1,5,9,\ldots,n-2\})$; $\sigma((i,j)) = 2$ if $(i,j) \in (\{1\} \times \{3,7,11,\ldots,n\}) \cup (\{2\} \times \{1,5,9,\ldots,n-2\}) \cup (\{3\} \times \{2,4,6,\ldots,n-1\})$. Hence, c is a modular 3-coloring of $C_3 \Box P_n$. See Table 2 for colors of the vertices and color sums of the vertices for $C_3 \Box P_n$.

color c			color sum σ		
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 2 1	0 1 2 1...0 1 2 1	0 1 2
1 0 2 0	1 0 2 0...1 0 2 0	1 0 2	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0

Table 2. $C_3 \Box P_n$ ($n \equiv 3 \mod 4$).

Case 3. $n \equiv 0 \mod 6$.

Define $c : V(C_3 \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1\} \times \{3,9,15,\ldots,n-3\}) \cup (\{1\} \times \{4,10,16,\ldots,n-2\}) \cup (\{1\} \times \{5,11,17,\ldots,n-1\}) \cup (\{2\} \times \{2,8,14,\ldots,n-4\})$; $c((i,j)) = 2$ if $(i,j) \in (\{2\} \times \{5,11,17,\ldots,n-1\}) \cup (\{3\} \times \{2,8,14,\ldots,n-4\}) \cup (\{3\} \times \{3,9,15,\ldots,n-3\}) \cup (\{3\} \times \{4,10,16,\ldots,n-2\})$; and $c((i,j)) = 0$ otherwise. Then,

$(\sigma((1,j)), \sigma((2,j)), \sigma((3,j))) = (0,1,2)$ if $j \in \{1,3,5,\ldots,n-1\}$;

$(\sigma((1,j)), \sigma((2,j)), \sigma((3,j))) = (1,2,0)$ if $j \in \{2,4,6,\ldots,n\}$.

Hence, c is a modular 3-coloring of $C_3 \square P_n$. See Table 3 for colors of the vertices and color sums of the vertices for $C_3 \square P_n$.

color c		color sum σ	
0 0 1 1 1 0	0 0 1 1 1 0. . . 0 0 1 1 1 0	0 1 0 1 0 1	0 1 0 1 0 1. . . 0 1 0 1 0 1
0 1 0 0 2 0	0 1 0 0 2 0. . . 0 1 0 0 2 0	1 2 1 2 1 2	1 2 1 2 1 2. . . 1 2 1 2 1 2
0 2 2 2 0 0	0 2 2 2 0 0. . . 0 2 2 2 0 0	2 0 2 0 2 0	2 0 2 0 2 0. . . 2 0 2 0 2 0

Table 3. $C_3 \square P_n$ ($n \equiv 0 \mod 6$).

Case 4. $n \in \{4, 10\}$.

See Tables 4 and 5 for colors of the vertices and color sums of the vertices for $C_3 \square P_4$ and $C_3 \square P_{10}$, respectively.

color c	color sum σ
0 1 1 1	1 0 1 0
1 0 0 2	2 1 2 1
2 2 2 0	0 2 0 2

Table 4. $C_3 \square P_4$.

color c	color sum σ
0 0 2 0 0 0 2 0 1 2	0 2 1 2 0 2 1 0 1 2
1 2 1 1 0 0 0 2 0 0	1 0 2 0 1 0 2 1 2 0
2 1 0 2 0 0 1 1 2 1	2 1 0 1 2 1 0 2 0 1

Table 5. $C_3 \square P_{10}$.

Finally, assume that $m \geq 4$ and $n \geq 3$. We consider the following 12 cases.

Case 1. $m \equiv 0 \mod 2$ and $n \equiv 0 \mod 4$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_2$ as follows: $c((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \ldots, m-1\} \times \{3, 7, 11, \ldots, n-1\}) \cup (\{2, 4, 6, \ldots, m\} \times \{2, 6, 10, \ldots, n-2\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \ldots, m\} \times \{1, 3, 5, \ldots, n-1\}) \cup (\{1, 3, 5, \ldots, m-1\} \times \{2, 4, 6, \ldots, n\})$; and $\sigma((i, j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See Table 6 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c		color sum σ	
0 0 1 0	0 0 1 0 . . . 0 0 1 0	0 1 0 1	0 1 0 1 . . . 0 1 0 1
0 1 0 0	0 1 0 0 . . . 0 1 0 0	1 0 1 0	1 0 1 0 . . . 1 0 1 0
0 0 1 0	0 0 1 0 . . . 0 0 1 0	0 1 0 1	0 1 0 1 . . . 0 1 0 1
0 1 0 0	0 1 0 0 . . . 0 1 0 0	1 0 1 0	1 0 1 0 . . . 1 0 1 0
$\vdots$ $\vdots$ $\vdots$ $\vdots$	$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$	$\vdots$ $\vdots$ $\vdots$ $\vdots$	$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$
0 0 1 0	0 0 1 0 . . . 0 0 1 0	0 1 0 1	0 1 0 1 . . . 0 1 0 1
0 1 0 0	0 1 0 0 . . . 0 1 0 0	1 0 1 0	1 0 1 0 . . . 1 0 1 0

Table 6. $C_m \square P_n$ ($m \equiv 0 \mod 2$ and $n \equiv 0 \mod 4$).

Case 2. $m \equiv 0 \mod 2$ and $n \equiv 2 \mod 4$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_2$ as follows: $c((i,j)) = 1$ if $(i,j) \in (\{1,3,5,\ldots,m-1\} \times \{2,6,10,\ldots,n\}) \cup (\{2,4,6,\ldots,m\} \times \{1,5,9,\ldots,n-1\})$; and $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,j)) = 1$ if $(i,j) \in (\{1,3,5\ldots,m-1\} \times \{1,3,5,\ldots,n-1\}) \cup (\{2,4,6\ldots,m\} \times \{2,4,6,\ldots,n\})$; and $\sigma((i,j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See Table 7 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c			color sum σ		
0 1 0 0	0 1 0 0...0 1 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1
0 1 0 0	0 1 0 0...0 1 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 1 0 0	0 1 0 0...0 1 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1

Table 7. $C_m \square P_n$ ($m \equiv 0 \mod 2$ and $n \equiv 2 \mod 4$).

Case 3. $m \equiv 2 \mod 4$ and $n \equiv 3 \mod 4$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_2$ as follows: $c((i,j)) = 1$ if $(i,j) \in (\{1,3,5,\ldots,m-1\} \times \{2,6,10,\ldots,n-1\}) \cup (\{2,4,6,\ldots,m\} \times \{1,5,9,\ldots,n-2\})$; and $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,j)) = 1$ if $(i,j) \in (\{1,3,5\ldots,m-1\} \times \{1,3,5,\ldots,n\}) \cup (\{2,4,6\ldots,m\} \times \{2,4,6,\ldots,n-1\})$; and $\sigma((i,j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See Table 8 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c			color sum σ		
0 1 0 0	0 1 0 0...0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 1 0 0	0 1 0 0...0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
0 1 0 0	0 1 0 0...0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0

Table 8. $C_m \square P_n$ ($m \equiv 2 \mod 4$ and $n \equiv 3 \mod 4$).

Case 4. $m \equiv 0 \mod 4$ and $n \equiv 1 \mod 2$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_2$ as follows:

For $n \equiv 1 \mod 4$, $c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-3\} \times \{1,5,9,\ldots,n\}) \cup (\{3,7,11,\ldots,m-1\} \times \{3,7,11,\ldots,n-2\})$; and $c((i,j)) = 0$ otherwise.

For $n \equiv 3 \mod 4$, $c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-3\} \times \{1,5,9,\ldots,n-2\}) \cup (\{3,7,11,\ldots,m-1\} \times \{3,7,11,\ldots,n\})$; and $c((i,j)) = 0$ otherwise.

Then, $\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m\} \times \{1,3,5,\ldots,n\}) \cup (\{1,3,5,\ldots,m-1\} \times \{2,4,6,\ldots,n-1\})$; and $\sigma((i,j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. For $n \equiv 1 \mod 4$ and $n \equiv 3 \mod 4$, respectively, see Tables 9 and 10 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

	color c			color sum σ	
1 0 0 0	1 0 0 0. . . 1 0 0 0	1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1
1 0 0 0	1 0 0 0. . . 1 0 0 0	1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮
1 0 0 0	1 0 0 0. . . 1 0 0 0	1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1

Table 9. $C_m \square P_n$ ($m \equiv 0 \mod 4$ and $n \equiv 1 \mod 4$).

	color c			color sum σ	
1 0 0 0	1 0 0 0. . . 1 0 0 0	1 0 0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0 0 1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1
1 0 0 0	1 0 0 0. . . 1 0 0 0	1 0 0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0 0 1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
1 0 0 0	1 0 0 0. . . 1 0 0 0	1 0 0	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1
0 0 1 0	0 0 1 0. . . 0 0 1 0	0 0 1	0 1 0 1	0 1 0 1. . . 0 1 0 1	0 1 0
0 0 0 0	0 0 0 0. . . 0 0 0 0	0 0 0	1 0 1 0	1 0 1 0. . . 1 0 1 0	1 0 1

Table 10. $C_m \square P_n$ ($m \equiv 0 \mod 4$ and $n \equiv 3 \mod 4$).

Case 5. $m \equiv 1 \mod 4$ and $n \equiv 1 \mod 4$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-4\} \times \{1,5,9,\ldots,n\}) \cup (\{3,7,11,\ldots,m-6\} \times \{3,7,11,\ldots,n-2\})$; $c((m-2,j)) = 2$ if $j \in \{3,7,11,\ldots,n-2\}$; $c((m-1,j)) = 1$ if $j \in \{3,7,11,\ldots,n-2\}$; $c((m,j)) = 2$ if $j \in \{1,5,9,\ldots,n\}$; and $c((i,j)) = 0$ otherwise. Then,

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1	2 1 0 1	2 1 0 1...2 1 0 1	2
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 2 0	1 0 2 0...1 0 2 0	1
0 0 2 0	0 0 2 0...0 0 2 0	0	0 2 1 2	0 2 1 2...0 2 1 2	0
0 0 1 0	0 0 1 0...0 0 1 0	0	2 1 2 1	2 1 2 1...2 1 2 1	2
2 0 0 0	2 0 0 0...2 0 0 0	2	1 2 1 2	1 2 1 2...1 2 1 2	1

Table 11. $C_m \Box P_n$ ($m \equiv 1 \mod 4$ and $n \equiv 1 \mod 4$).

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3,5,7,\ldots,m-4\} \times \{1,3,5,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n\}) \cup (\{3,5,7,\ldots,m-4\} \times \{2,4,6,\ldots,n-1\})$;

$\sigma((1,j)) = 0$ if $j \in \{3,7,11,\ldots,n-2\}$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((1,j)) = 2$ if $j \in \{1,5,9,\ldots,n\}$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{1,5,9,\ldots,n\}$;

$\sigma((m-3,j)) = 2$ if $j \in \{3,7,11,\ldots,n-2\}$;

$\sigma((m-2,j)) = 0$ if $j \in \{1,5,9,\ldots,n\}$;

$\sigma((m-2,j)) = 1$ if $j \in \{3,7,11,\ldots,n-2\}$;

$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((m-1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n\}$;

$\sigma((m,j)) = 1$ if $j \in \{1,3,5,\ldots,n\}$;

$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$.

Hence, c is a modular 3-coloring of $C_m \Box P_n$. See Table 11 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$.

Case 6. $m \equiv 1 \mod 4$ and $n \equiv 3 \mod 4$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-4\} \times \{1,5,9,\ldots,n-2\}) \cup (\{3,7,11,\ldots,m-6\} \times \{3,7,11,\ldots,n\})$; $c((m-2,j)) = 2$ if $j \in \{3,7,11,\ldots,n\}$; $c((m-1,j)) = 1$ if $j \in \{3,7,11,\ldots,n\}$; $c((m,j)) = 2$ if $j \in \{1,5,9,\ldots,n-2\}$; and $c((i,j)) = 0$ otherwise. Then,

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	2 1 0 1	2 1 0 1...2 1 0 1	2 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 2 0	1 0 2 0...1 0 2 0	1 0 2
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2	0 2 1 2	0 2 1 2...0 2 1 2	0 2 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	2 1 2 1	2 1 2 1...2 1 2 1	2 1 2
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	1 2 1 2	1 2 1 2...1 2 1 2	1 2 1

Table 12. $C_m \Box P_n$ ($m \equiv 1 \mod 4$ and $n \equiv 3 \mod 4$).

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3,5,7,\ldots,m-4\} \times \{1,3,5,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n\}) \cup (\{3,5,7,\ldots,m-4\} \times \{2,4,6,\ldots,n-1\})$;

$\sigma((1,j)) = 0$ if $j \in \{3,7,11,\ldots,n\}$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((1,j)) = 2$ if $j \in \{1,5,9,\ldots,n-2\}$;
$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-3,j)) = 1$ if $j \in \{1,5,9,\ldots,n-2\}$;
$\sigma((m-3,j)) = 2$ if $j \in \{3,7,11,\ldots,n\}$;
$\sigma((m-2,j)) = 0$ if $j \in \{1,5,9,\ldots,n-2\}$;
$\sigma((m-2,j)) = 1$ if $j \in \{3,7,11,\ldots,n\}$;
$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m,j)) = 1$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 12 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

Case 7. $m \equiv 3 \mod 4$ and $n \equiv 1 \mod 4$.

Define $c : V(C_m \square P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{1,5,9,\ldots,n\}) \cup (\{3,7,11,\ldots,m-4\} \times \{3,7,11,\ldots,n-2\})$; $c((m-2,j)) = 2$ if $j \in \{1,5,9,\ldots,n\}$; $c((m,j)) = 2$ if $j \in \{3,7,11,\ldots,n-2\}$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3,5,7,\ldots,m-4\} \times \{1,3,5,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n\}) \cup (\{3,5,7,\ldots,m-4\} \times \{2,4,6,\ldots,n-1\})$;

$\sigma((1,j)) = 0$ if $j \in \{1,5,9,\ldots,n\}$;
$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((1,j)) = 2$ if $j \in \{3,7,11,\ldots,n-2\}$;
$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-3,j)) = 1$ if $j \in \{3,7,11,\ldots,n-2\}$;
$\sigma((m-3,j)) = 2$ if $j \in \{1,5,9,\ldots,n\}$;
$\sigma((m-2,j)) = 0$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m,j)) = 0$ if $j \in \{3,7,11,\ldots,n-2\}$;
$\sigma((m,j)) = 1$ if $j \in \{1,5,9,\ldots,n\}$;
$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 13 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 2 1	0 1 2 1...0 1 2 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	2 0 1 0	2 0 1 0...2 0 1 0	2
2 0 0 0	2 0 0 0...2 0 0 0	2	0 2 0 2	0 2 0 2...0 2 0 2	0
0 0 0 0	0 0 0 0...0 0 0 0	0	2 0 2 0	2 0 2 0...2 0 2 0	2
0 0 2 0	0 0 2 0...0 0 2 0	0	1 2 0 2	1 2 0 2...1 2 0 2	1

Table 13. $C_m \Box P_n$ ($m \equiv 3 \mod 4$ and $n \equiv 1 \mod 4$).

Case 8. $m \equiv 3 \mod 4$ and $n \equiv 3 \mod 4$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{1,5,9,\ldots,n-2\}) \cup (\{3,7,11,\ldots, m-4\} \times \{3,7,11,\ldots,n\})$; $c((m-2,j)) = 2$ if $j \in \{1,5,9,\ldots,n-2\}$; $c((m,j)) = 2$ if $j \in \{3,7,11,\ldots,n\}$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-1\}) \cup (\{3,5,7,\ldots,m-4\} \times \{1,3,5,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n\}) \cup (\{3,5,7,\ldots, m-4\} \times \{2,4,6,\ldots, n-1\})$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((1,j)) = 0$ if $j \in \{1,5,9,\ldots,n-2\}$;

$\sigma((1,j)) = 2$ if $j \in \{3,7,11,\ldots,n\}$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{3,7,11,\ldots,n\}$;

$\sigma((m-3,j)) = 2$ if $j \in \{1,5,9,\ldots,n-2\}$;
$\sigma((m-2,j)) = 0$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 0$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n\}$;
$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-1\}$;
$\sigma((m,j)) = 0$ if $j \in \{3,7,11,\ldots,n\}$,
$\sigma((m,j)) = 1$ if $j \in \{1,5,9,\ldots,n-2\}$.

Hence, c is a modular 3-coloring of $C_m \Box P_n$. See Table 14 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$.

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 2 1	0 1 2 1...0 1 2 1	0 1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0

Table 14. $C_m \Box P_n$ ($m \equiv 3 \mod 4$ and $n \equiv 3 \mod 4$).

Case 9. $m \equiv 1 \mod 4$ and $n \equiv 0 \mod 4$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-4\} \times \{1,5,9,\ldots,n-3\}) \cup (\{3,7,11,\ldots,m-6\} \times \{3,7,11,\ldots,n-1\}) \cup \{(m-2,n-1)\} \cup (\{m-1\} \times \{3,7,11,\ldots,n-5\})$; $c((i,j)) = 2$ if $(i,j) \in (\{1,5,9,\ldots,m-4\}$

$\times \{n\}) \cup (\{m-2\} \times \{3, 7, 11, \ldots, n-5\}) \cup (\{m\} \times \{1, 5, 9, \ldots, n-3\})$; and $c((i,j)) = 0$ otherwise. Then,

color c				color sum σ			
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0 2	2 1 0 1	2 1 0 1...	2 1 0 1	2 1 2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 2 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0 0	1 0 2 0	1 0 2 0...	1 0 2 0	1 0 1 2
0 0 2 0	0 0 2 0...	0 0 2 0	0 0 1 0	0 2 1 2	0 2 1 2...	0 2 1 2	0 1 0 1
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 0 0	2 1 2 1	2 1 2 1...	2 1 2 1	2 0 1 0
2 0 0 0	2 0 0 0...	2 0 0 0	2 0 0 0	1 2 1 2	1 2 1 2...	1 2 1 2	1 2 0 2

Table 15. $C_m \Box P_n$ ($m \equiv 1 \mod 4$ and $n \equiv 0 \mod 4$).

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-2\}) \cup (\{3,7,11,\ldots,m-6\} \times \{1,3,5,\ldots, n-1\}) \cup (\{5,9,13,\ldots,m-4\} \times \{1,3,5,\ldots,n-3\}) \cup (\{5,9,13,\ldots,m-4\} \times \{n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n-1\}) \cup (\{3,7,11,\ldots,m-6\} \times \{2,4,6,\ldots,n\}) \cup (\{5,9,13,\ldots,m-4\} \times \{2,4,6,\ldots,n-2\})$;

$\sigma((i,n-1)) = 2$ if $i \in \{5,9,13,\ldots,m-4\}$;

$\sigma((i,n)) = 2$ if $i \in \{2,4,6,\ldots,m-5\}$;

$\sigma((1,j)) = 0$ if $j \in \{3,7,11,\ldots,n-5\}$;

$\sigma((1,n)) = 0$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((1,j)) = 2$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((1,n-1)) = 2$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((m-3,n-1)) = 1$;

$\sigma((m-3,j)) = 2$ if $j \in \{3,7,11,\ldots,n-5\}$;

$\sigma((m-3,n)) = 2$;

$\sigma((m-2,j)) = 0$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((m-2,n-1)) = 0$;

$\sigma((m-2,j)) = 1$ if $j \in \{3,7,11,\ldots,n-5\}$;

$\sigma((m-2,n-2)) = \sigma((m-2,n)) = 1$;

$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n-4\}$;

$\sigma((m-1,n-2)) = \sigma((m-1,n)) = 0$;

$\sigma((m-1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-4\}$;

$\sigma((m-1,n-1)) = 1$;

$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n-3\}$;

$\sigma((m,n-1)) = 0$;

$\sigma((m,j)) = 1$ if $j \in \{1,3,5,\ldots,n-3\}$;

$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n\}$.

Hence, c is a modular 3-coloring of $C_m \Box P_n$. See Table 15 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$.

Case 10. $m \equiv 1 \mod 4$ and $n \equiv 2 \mod 4$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-4\} \times \{1,5,9,\ldots,n-1\}) \cup (\{3,7,11,\ldots,m-6\} \times \{3,7,11,\ldots,n-3\}) \cup (\{m-1\} \times \{3,7,11,\ldots,n-3\})$; $c((i,j)) = 2$ if $(i,j) \in (\{3,7,11,\ldots,m-6\} \times \{n\}) \cup (\{m-2\} \times \{3,7,11,\ldots,n-3\}) \cup \{(m-1,n)\} \cup (\{m\} \times \{1,5,9,\ldots,n-5\})$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{2,4,6,\ldots,n-2\}) \cup (\{3,7,11,\ldots,m-6\} \times \{1,3,5,\ldots,n-3\}) \cup (\{3,7,11,\ldots,m-6\} \times \{n\}) \cup (\{5,9,13,\ldots,m-4\} \times \{1,3,5,\ldots,n-1\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,4,6,\ldots,m-5\} \times \{1,3,5,\ldots,n-1\}) \cup (\{3,7,11,\ldots,m-6\} \times \{2,4,6,\ldots,n-2\}) \cup (\{5,9,13,\ldots,m-4\} \times \{2,4,6,\ldots,n\})$;

$\sigma((i,n-1)) = 2$ if $i \in \{3,7,11,\ldots,m-6\}$;

$\sigma((i,n)) = 2$ if $i \in \{2,4,6,\ldots,m-5\}$;

$\sigma((1,j)) = 0$ if $j \in \{3,7,11,\ldots,n-3\}$;

$\sigma((1,n-1)) = 0$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((1,j)) = 2$ if $j \in \{1,5,9,\ldots,n-5\}$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{1,5,9,\ldots,n-1\}$;

$\sigma((m-3,j)) = 2$ if $j \in \{3,7,11,\ldots,n-3\}$;

$\sigma((m-2,j)) = 0$ if $j \in \{1,5,9,\ldots,n-1\}$;

$\sigma((m-2,j)) = 1$ if $j \in \{3,7,11,\ldots,n-3\}$;
$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n\}$;
$\sigma((m-1,n)) = 0$;
$\sigma((m-1,j)) = 1$ if $j \in \{2,4,6,\ldots,n-2\}$;
$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n-1\}$;
$\sigma((m,n-2)) = 0$;
$\sigma((m,j)) = 1$ if $j \in \{1,3,5,\ldots,n-1\}$;
$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-4\}$;
$\sigma((m,n)) = 2$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 16 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0	2 1 0 1	2 1 0 1...2 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 2	0 1 0 1	0 1 0 1...0 1 0 1	2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 2	0 1 0 1	0 1 0 1...0 1 0 1	2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 2	0 1 0 1	0 1 0 1...0 1 0 1	2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0	1 0 2 0	1 0 2 0...1 0 2 0	1 0
0 0 2 0	0 0 2 0...0 0 2 0	0 0	0 2 1 2	0 2 1 2...0 2 1 2	0 2
0 0 1 0	0 0 1 0...0 0 1 0	0 2	2 1 2 1	2 1 2 1...2 1 2 1	2 0
2 0 0 0	2 0 0 0...2 0 0 0	0 0	1 2 1 2	1 2 1 2...1 2 1 0	1 2

Table 16. $C_m \square P_n$ ($m \equiv 1 \mod 4$ and $n \equiv 2 \mod 4$).

Case 11. $m \equiv 3 \mod 4$ and $n \equiv 0 \mod 4$.

Subcase 11.1. $m \geq 7$ and $n = 4$.

Define $c : V(C_m \square P_4) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{3,7,11,\ldots,m-4\} \times \{1\}) \cup (\{1,5,9,\ldots,m-6\} \times \{3\}) \cup (\{4,8,12,\ldots,m-7\}$

$\times \{4\}) \cup \{(m-2,4)\}$; $c((i,j)) = 2$ if $(i,j) \in (\{2,6,10,\ldots,m-5\} \times \{1\}) \cup \{(m-3,4),(m-1,2)\}$ and $c((i,j)) = 0$ otherwise. Then,

$(\sigma((1,1)), \sigma((1,2)), \sigma((1,3)), \sigma((1,4))) = (2,1,0,1)$;

$(\sigma((i,1)), \sigma((i,2)), \sigma((i,3)), \sigma((i,4))) = (1,2,1,0)$ if $i \in \{2,6,10,\ldots,m-5\}$;

$(\sigma((i,1)), \sigma((i,2)), \sigma((i,3)), \sigma((i,4))) = (2,1,0,1)$ if $i \in \{3,7,11,\ldots,m-8\}$;

$(\sigma((i,1)), \sigma((i,2)), \sigma((i,3)), \sigma((i,4))) = (1,0,2,0)$ if $i \in \{4,8,12,\ldots,m-7\}$;

$(\sigma((i,1)), \sigma((i,2)), \sigma((i,3)), \sigma((i,4))) = (2,1,0,2)$ if $i \in \{5,9,13,\ldots,m-6\}$;

$(\sigma((m-4,1)), \sigma((m-4,2)), \sigma((m-4,3)), \sigma((m-4,4))) = (2,1,0,2)$;

$(\sigma((m-3,1)), \sigma((m-3,2)), \sigma((m-3,3)), \sigma((m-3,4))) = (1,0,2,1)$;

$(\sigma((m-2,1)), \sigma((m-2,2)), \sigma((m-2,3)), \sigma((m-2,4))) = (0,2,1,2)$;

$(\sigma((m-1,1)), \sigma((m-1,2)), \sigma((m-1,3)), \sigma((m-1,4))) = (2,0,2,1)$;

$(\sigma((m,1)), \sigma((m,2)), \sigma((m,3)), \sigma((m,4))) = (0,2,1,0)$.

Hence, c is a modular 3-coloring of $C_m \Box P_4$. See Table 17 for colors of the vertices and color sums of the vertices for $C_m \Box P_4$.

color c	color sum σ
0 0 1 0	2 1 0 1
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 2
0 0 0 2	1 0 2 1
0 0 0 1	0 2 1 2
0 2 0 0	2 0 2 1
0 0 0 0	0 2 1 0

Table 17. $C_m \Box P_4$ ($m \equiv 3 \mod 4$).

Subcase 11.2. $m \geq 7$ and $n \geq 8$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{3,7,11,\ldots,n-1\}) \cup (\{3,7,11,\ldots,m-4\} \times \{1,5,9,\ldots,n-3\}) \cup (\{4,8,12,\ldots,m-7\} \times \{n\}) \cup \{(m-2,\,n)\}$; $c((i,j)) = 2$ if $(i,j) \in \{(1,n-4),(m-3,n),(m-1,n-2)\} \cup (\{m-2\} \times \{3,7,11,\ldots,n-5\}) \cup (\{m\} \times \{1,5,9,\ldots,n-7\}$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{3,5,7,\ldots,m-6\} \times \{1,3,5,\ldots,n-1\}) \cup (\{4,6,8,\ldots,m-5\} \times \{2,4,6,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{3,7,11,\ldots,m-8\} \times \{2,4,6,\ldots,n\}) \cup (\{5,9,13,\ldots,m-6\} \times \{2,4,6,\ldots,n-2\}) \cup (\{6,10,14,\ldots,m-5\} \times \{1,3,5,\ldots,\,n-1\}) \cup (\{4,8,12\ldots,m-7\} \times \{1,3,5,\ldots,n-3\})$;

$\sigma((i,n-1)) = 2$ if $i \in \{4,8,12,\ldots,m-7\}$;

$\sigma((i,n)) = 2$ if $i \in \{5,9,13,\ldots,m-6\}$;

$\sigma((1,j)) = 0$ if $j \in \{3,7,11,\ldots,n-9\}$;

$\sigma((1,n-1)) = 0$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((1,j)) = 2$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((1,n-5)) = 2$;

$\sigma((2,j)) = 0$ if $j \in \{2,4,6,\ldots,n-6\} \cup \{n-2,n\}$;

$\sigma((2,j)) = 1$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((2,n-4)) = 2$;

$\sigma((m-4,j)) = 0$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((m-4,j)) = 1$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-4,n)) = 2$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((m-3,n)) = 1$;

$\sigma((m-3,j)) = 2$ if $j \in \{3,7,11,\ldots,n-1\}$;

$\sigma((m-2,j)) = 0$ if $j \in \{1,3,5,\ldots,n-3\}$;

$\sigma((m-2,n-1)) = 1$;

$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((m-1,j)) = 0$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-1,n)) = 1$;

$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((m,j)) = 0$ if $j \in \{1,5,9,\ldots,n-3\}$;

$\sigma((m,n)) = 0$;

$\sigma((m,j)) = 1$ if $j \in \{3,7,11,\ldots,n-1\}$;

$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-2\}$.

Hence, c is a modular 3-coloring of $C_m \Box P_n$. See Table 18 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$.

color c				color sum σ			
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 2	0 0 1 0	2 1 0 1	2 1 0 1...2 1 0 1	2 1 2 1	2 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 2	1 0 2 0	1 0 2 0...1 0 2 0	1 0 2 0	1 0 2 1
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2 0	0 0 0 1	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0 2	0 2 1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 2 0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2 0	2 0 2 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0 0	0 0 0 0	0 2 1 2	0 2 1 2...0 2 1 2	0 2 1 2	0 2 1 0

Table 18. $C_m \Box P_n$ ($m \equiv 3 \mod 4$, $m \geq 7$ and $n \equiv 0 \mod 4$, $n \geq 8$).

Case 12. $m \equiv 3 \mod 4$ and $n \equiv 2 \mod 4$.

Subcase 12.1. $m \geq 7$ and $n \geq 10$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{1,5,9,\ldots,n-1\}) \cup (\{3,7,11,\ldots,m-4\} \times \{3,7,11,\ldots,n-3\}) \cup (\{4,8,12,\ldots,m-7\} \times \{n\}) \cup \{(m-2,n)\}$; $c((i,j)) = 2$ if $(i,j) \in \{(1,n-4),(m-3,n)\} \cup (\{m-2\} \times \{1,5,9,\ldots,n-5\}) \cup \{(m-1,n-2)\} \cup (\{m\} \times \{3,7,11,\ldots,n-7\})$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{3,7,11,\ldots,m-8\} \times \{1,3,5,\ldots,n-1\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{3,7,11,\ldots,m-8\} \times \{2,4,6,\ldots,n\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{2,4,6,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{1,3,5,\ldots,n-3\})$;

$\sigma((i,j)) = 2$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{n-1\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{1,3,5,\ldots,n-1\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{2,4,6,\ldots,n-2\})$;

$\sigma((i,j)) = 2$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{n\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{6,10,14,\ldots,m-5\} \times \{2,4,6,\ldots,n\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{6,10,14,\ldots,m-5\} \times \{1,3,5,\ldots,n-1\})$;

$\sigma((1,j)) = 0$ if $j \in \{1,5,9,\ldots,n-9\}$;

$\sigma((1,n-1)) = 0$;

$\sigma((1,j)) = 1$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((1,j)) = 2$ if $j \in \{3,7,11,\ldots,n-3\}$;

$\sigma((1,n-5)) = 2$;

$\sigma((2,j)) = 0$ if $j \in \{2,4,6,\ldots,n-6\}$;

$\sigma((2,n-2)) = \sigma((2,n)) = 0$;

$\sigma((2,j)) = 1$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((2,n-4)) = 2$;

$\sigma((m-4,j)) = 0$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((m-4,j)) = 1$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-4,n)) = 2$;

$\sigma((m-3,j)) = 0$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-3,j)) = 1$ if $j \in \{3,7,11,\ldots,n-3\}$;

$\sigma((m-3,n)) = 1$;

$\sigma((m-3,j)) = 2$ if $j \in \{1,5,9,\ldots,n-1\}$;

$\sigma((m-2,j)) = 0$ if $j \in \{1,3,5,\ldots,n-3\}$;

$\sigma((m-2,n-1)) = 1$;

$\sigma((m-2,j)) = 2$ if $j \in \{2,4,6,\ldots,n\}$;

$\sigma((m-1,j)) = 0$ if $j \in \{2,4,6,\ldots,n-2\}$;

$\sigma((m-1,n)) = 1$;

$\sigma((m-1,j)) = 2$ if $j \in \{1,3,5,\ldots,n-1\}$;

$\sigma((m,j)) = 0$ if $j \in \{3,7,11,\ldots,n-3\}$;

$\sigma((m,n)) = 0$;

$\sigma((m,j)) = 1$ if $j \in \{1,5,9,\ldots,n-1\}$;

$\sigma((m,j)) = 2$ if $j \in \{2,4,6,\ldots,n-2\}$.

Hence, c is a modular 3-coloring of $C_m \Box P_n$. See Table 19 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$.

color c				color sum σ			
1 0 0 0	1 0 0 0...1 0 0 0	1 2 0 0	1 0	0 1 2 1	0 1 2 1...0 1 2 1	2 1 2 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 2	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1 0	2 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0 0	0 1	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0 2	1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 2	0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2 0	2 1
0 0 2 0	0 0 2 0...0 0 2 0	0 0 0 0	0 0	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0 2	1 0

Table 19. $C_m \Box P_n$ ($m \equiv 3 \mod 4$, $m \geq 7$ and $n \equiv 2 \mod 4$, $n \geq 10$).

Subcase 12.2. $m \geq 7$ and $n = 6$.

Define $c : V(C_m \Box P_6) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{1,5\}) \cup (\{3,7,11,\ldots,m-4\} \times \{3\}) \cup (\{4,8,12,\ldots,m-7\} \times \{6\}) \cup \{(m-2,6)\}$; $c((i,j)) = 2$ if $(i,j) \in (\{1,5,9,\ldots,m-6\} \times \{2\}) \cup \{(m-3,6),(m-2,1),(m-1,4)\}$; and $c((i,j)) = 0$ otherwise. Then,

$\sigma((i,j)) = 0$ if $(i,j) \in (\{2,6,10,\ldots,m-5\} \times \{4,6\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{2,6,10,\ldots,m-5\} \times \{1,3,5\})$;

$\sigma((i,j)) = 2$ if $(i,j) \in (\{2,6,10,\ldots,m-5\} \times \{2\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{3,7,11,\ldots,m-8\} \times \{1,3,5\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{3,7,11,\ldots,m-8\} \times \{2,4,6\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{4,6\})$;

$\sigma((i,j)) = 1$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{1,3\})$;

$\sigma((i,j)) = 2$ if $(i,j) \in (\{4,8,12,\ldots,m-7\} \times \{2,5\})$;

$\sigma((i,j)) = 0$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{5\})$;

color c	color sum σ
1 2 0 0 1 0	2 1 2 1 0 1
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 2
0 0 0 0 0 2	2 0 1 0 2 1
2 0 0 0 0 1	0 2 0 2 1 2
0 0 0 2 0 0	2 0 2 0 2 1
0 0 0 0 0 0	1 2 0 2 1 0

Table 20. $C_m \square P_6$ ($m \equiv 3 \mod 4$, $m \geq 7$).

$\sigma((i,j)) = 1$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{2,4\})$;
$\sigma((i,j)) = 2$ if $(i,j) \in (\{5,9,13,\ldots,m-6\} \times \{1,3,6\})$;
$\sigma((1,5)) = 0$;
$\sigma((1,2)) = \sigma((1,4)) = \sigma((1,6)) = 1$;
$\sigma((1,1)) = \sigma((1,3)) = 2$;
$\sigma((m-4,1)) = \sigma((m-4,3)) = \sigma((m-4,5)) = 0$;
$\sigma((m-4,2)) = \sigma((m-4,4)) = 1$;
$\sigma((m-4,6)) = 2$;
$\sigma((m-3,2)) = \sigma((m-3,4)) = 0$;
$\sigma((m-3,3)) = \sigma((m-3,6)) = 1$;
$\sigma((m-3,1)) = \sigma((m-3,5)) = 2$;
$\sigma((m-2,1)) = \sigma((m-2,3)) = 0$;
$\sigma((m-2,5)) = 1$;
$\sigma((m-2,2)) = \sigma((m-2,4)) = \sigma((m-2,6)) = 2$;

$\sigma((m-1,2)) = \sigma((m-1,4)) = 0;$
$\sigma((m-1,6)) = 1;$
$\sigma((m-1,1)) = \sigma((m-1,3)) = \sigma((m-1,5)) = 2;$
$\sigma((m,3)) = \sigma((m,6)) = 0;$
$\sigma((m,1)) = \sigma((m,5)) = 1;$
$\sigma((m,2)) = \sigma((m,4)) = 2.$

Hence, c is a modular 3-coloring of $C_m \Box P_6$. See Table 20 for colors of the vertices and color sums of the vertices for $C_m \Box P_6$.

This completes the proof. □

Proof. (iii): By hypothesis, $m \equiv 2 \mod 4$ and $n \equiv 1 \mod 4$.

Define $c : V(C_m \Box P_n) \to \mathbb{Z}_3$ as follows:

$c((i,j)) = 1$ if $(i,j) \in (\{1,5,9\ldots,m-5\} \times \{1,5,9,\ldots,n\}) \cup (\{3,7,11\ldots,m-3\} \times \{3,7,11,\ldots,n-2\}) \cup (\{m-1\} \times \{1,3,5,\ldots,n\})$; and $c((i,j)) = 0$ otherwise. Then,

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
1 0 0 0	1 0 0 0...1 0 0 0	1	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 1 0	1 0 1 0...1 0 1 0	1
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 0 1	0 1 0 1...0 1 0 1	0
0 0 0 0	0 0 0 0...0 0 0 0	0	1 0 2 0	1 0 2 0...1 0 2 0	1
1 0 1 0	1 0 1 0...1 0 1 0	1	0 2 0 2	0 2 0 2...0 2 0 2	0
0 0 0 0	0 0 0 0...0 0 0 0	0	2 0 1 0	2 0 1 0...2 0 1 0	2

Table 21. $C_m \Box P_n$ ($m \equiv 2 \mod 4$ and $n \equiv 1 \mod 4$).

$\sigma((i,j)) = 0$ if $(i,j) \in (\{1,3,5,\ldots,m-1\} \times \{1,3,5,\ldots,n\}) \cup$
$(\{2,4,6,\ldots,m\} \times \{2,4,6,\ldots,n-1\});$

$\sigma((i,j)) = 1$ if $(i,j) \in (\{1,3,5,\ldots,m-3\} \times \{2,4,6,\ldots,n-1\}) \cup$
$(\{2,4,6,\ldots,m-4\} \times \{1,3,5,\ldots,n\}) \cup (\{m-2\} \times \{1,5,9,\ldots,n\}) \cup (\{m\} \times \{3,7,9,\ldots,n-2\})$;
$\sigma((i,j)) = 2$ if $(i,j) \in (\{m-2\} \times \{3,7,11,\ldots,n-2\}) \cup$
$(\{m-1\} \times \{2,4,6,\ldots,n-1\}) \cup (\{m\} \times \{1,5,9,\ldots,n\})$.
See Table 21 for colors of the vertices and color sums of the vertices for $C_m \Box P_n$. □

Proof. (iv): By (iii), $mc(C_6 \Box P_n) \leq 3$. Hence, we have to show that $mc(C_6 \Box P_n) \geq 3$. Assume, to the contrary, that there exists a modular 2-coloring c of $C_6 \Box P_n$. By the symmetry of the graph $C_6 \Box P_n$, we may assume that $\sigma((i,j)) = 0$ for (i,j) such that i and j are of different parity and $\sigma((i,j)) = 1$ for (i,j) such that i and j are of same parity. $\sigma((1,1)) = 1$ implies that $c((1,2)) = c((2,1)) = c((6,1)) = 1$, or $c((1,2)) = 1$ and $c((2,1)) = c((6,1)) = 0$, or $c((2,1)) = 1$ and $c((1,2)) = c((6,1)) = 0$, or $c((6,1)) = 1$ and $c((1,2)) = c((2,1)) = 0$. The case $c((6,1)) = 1$ and $c((1,2)) = c((2,1)) = 0$ is similar to the case $c((2,1)) = 1$ and $c((1,2)) = c((6,1)) = 0$.

Case 1. $c((1,2)) = c((2,1)) = c((6,1)) = 1$.

$\sigma((3,1)) = 1$ implies that $c((3,2)) = c((4,1))$.

Subcase 1.1. $c((3,2)) = c((4,1)) = 0$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 0$. Hence,
$(c((2,1)), c((4,1)), c((6,1))) = (1,0,1)$ and $(c((1,2)), c((3,2)), c((5,2))) = (1,0,0)$.
$(\sigma((2,2)), \sigma((4,2)), \sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)), c((4,3)), c((6,3))) = (1,1,1)$.
$(\sigma((1,3)), \sigma((3,3)), \sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)), c((3,4)), c((5,4))) = (0,1,1)$.
$(\sigma((2,4)), \sigma((4,4)), \sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)), c((4,5)), c((6,5))) = (1,0,1)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
$(\sigma((1,5)), \sigma((3,5)), \sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)), c((3,6)), c((5,6))) = (1,1,1)$.
$(\sigma((2,6)), \sigma((4,6)), \sigma((6,6))) = (1,1,1) \Rightarrow (c((2,7)), c((4,7)), c((6,7))) = (0,1,0)$.
$(\sigma((1,7)), \sigma((3,7)), \sigma((5,7))) = (1,1,1) \Rightarrow (c((1,8)), c((3,8)), c((5,8))) = (0,1,1)$.
$(\sigma((2,8)), \sigma((4,8)), \sigma((6,8))) = (1,1,1) \Rightarrow (c((2,9)), c((4,9)), c((6,9))) = (0,0,0)$.

If $n = 9$, then $\sigma((1,9)) = 0$, a contradiction. Hence assume that $n \geq 13$.
$(\sigma((1,9)), \sigma((3,9)), \sigma((5,9))) = (1,1,1) \Rightarrow (c((1,10)), c((3,10)), c((5,10))) = (1,0,0)$.
$(\sigma((2,10)), \sigma((4,10)), \sigma((6,10))) = (1,1,1) \Rightarrow (c((2,11)), c((4,11)), c((6,11))) = (0,1,0)$.
$(\sigma((1,11)), \sigma((3,11)), \sigma((5,11))) = (1,1,1) \Rightarrow (c((1,12)), c((3,12)), c((5,12))) = (0,0,0)$.
$(\sigma((2,12)), \sigma((4,12)), \sigma((6,12))) = (1,1,1) \Rightarrow (c((2,13)), c((4,13)), c((6,13))) = (1,0,1)$.

If $n = 13$, then $\sigma((1,13)) = 0$, a contradiction. Hence assume that $n \geq 17$.
$(\sigma((1,13)), \sigma((3,13)), \sigma((5,13))) = (1,1,1) \Rightarrow (c((1,14)), c((3,14)), c((5,14))) = (1,0,0)$.

Observe that
$(c((2,13)), c((4,13)), c((6,13))) = (1,0,1) = (c((2,1)), c((4,1)), c((6,1)))$ and
$(c((1,14)), c((3,14)), c((5,14))) = (1,0,0) = (c((1,2)), c((3,2)), c((5,2)))$.
This implies that
$(c((2,j)), c((4,j)), c((6,j))) = (c((2,j+12)), c((4,j+12)), c((6,j+12)))$ for odd $j \geq 3$
and

$(c((1,j)),c((3,j)),c((5,j))) = (c((1,j+12)),c((3,j+12)),c((5,j+12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 1.2. $c((3,2)) = c((4,1)) = 1$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 1$. Hence,

$(c((2,1)),c((4,1)),c((6,1))) = (1,1,1)$ and $(c((1,2)),c((3,2)),c((5,2))) = (1,1,1)$.

$(\sigma((2,2)),\sigma((4,2)),\sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)),c((4,3)),c((6,3))) = (0,0,0)$.

$(\sigma((1,3)),\sigma((3,3)),\sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)),c((3,4)),c((5,4))) = (0,0,0)$.

$(\sigma((2,4)),\sigma((4,4)),\sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)),c((4,5)),c((6,5))) = (1,1,1)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$(\sigma((1,5)),\sigma((3,5)),\sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)),c((3,6)),c((5,6))) = (1,1,1)$.

Observe that

$(c((2,5)),c((4,5)),c((6,5))) = (1,1,1) = (c((2,1)),c((4,1)),c((6,1)))$ and

$(c((1,6)),c((3,6)),c((5,6))) = (1,1,1) = (c((1,2)),c((3,2)),c((5,2)))$.

This implies that

$(c((2,j)),c((4,j)),c((6,j))) = (c((2,j+4)),c((4,j+4)),c((6,j+4)))$ for odd $j \geq 3$

and

$(c((1,j)),c((3,j)),c((5,j))) = (c((1,j+4)),c((3,j+4),c((5,j+4)))$ for even $j \geq 4$.

As we have obtained a contradiction for $n = 5$, we have a contradiction for $n = 9$, and so on.

Case 2. $c((1,2)) = 1$ and $c((2,1)) = c((6,1)) = 0$.

$\sigma((3,1)) = 1$ implies that $c((3,2)) \neq c((4,1))$.

Subcase 2.1. $c((3,2)) = 0$ and $c((4,1)) = 1$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 0$. Hence,

$(c((2,1)),c((4,1)),c((6,1))) = (0,1,0)$ and $(c((1,2)),c((3,2)),c((5,2))) = (1,0,0)$.

$(\sigma((2,2)),\sigma((4,2)),\sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)),c((4,3)),c((6,3))) = (0,0,0)$.

$(\sigma((1,3)),\sigma((3,3)),\sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)),c((3,4)),c((5,4))) = (0,1,1)$.

$(\sigma((2,4)),\sigma((4,4)),\sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)),c((4,5)),c((6,5))) = (0,1,0)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$(\sigma((1,5)),\sigma((3,5)),\sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)),c((3,6)),c((5,6))) = (1,1,1)$.

$(\sigma((2,6)),\sigma((4,6)),\sigma((6,6))) = (1,1,1) \Rightarrow (c((2,7)),c((4,7)),c((6,7))) = (1,0,1)$.

$(\sigma((1,7)),\sigma((3,7)),\sigma((5,7))) = (1,1,1) \Rightarrow (c((1,8)),c((3,8)),c((5,8))) = (0,1,1)$.

$(\sigma((2,8)),\sigma((4,8)),\sigma((6,8))) = (1,1,1) \Rightarrow (c((2,9)),c((4,9)),c((6,9))) = (1,1,1)$.

If $n = 9$, then $\sigma((1,9)) = 0$, a contradiction. Hence assume that $n \geq 13$.

$(\sigma((1,9)),\sigma((3,9)),\sigma((5,9))) = (1,1,1) \Rightarrow (c((1,10)),c((3,10)),c((5,10))) = (1,0,0)$.

$(\sigma((2,10)),\sigma((4,10)),\sigma((6,10))) = (1,1,1) \Rightarrow (c((2,11)),c((4,11)),c((6,11))) = (1,0,1)$.

$(\sigma((1,11)),\sigma((3,11)),\sigma((5,11))) = (1,1,1) \Rightarrow (c((1,12)),c((3,12)),c((5,12))) = (0,0,0)$.

$(\sigma((2,12)),\sigma((4,12)),\sigma((6,12))) = (1,1,1) \Rightarrow (c((2,13)),c((4,13)),c((6,13))) = (0,1,0)$.

If $n = 13$, then $\sigma((1,13)) = 0$, a contradiction. Hence assume that $n \geq 17$.

$(\sigma((1,13)),\sigma((3,13)),\sigma((5,13))) = (1,1,1) \Rightarrow (c((1,14)),c((3,14)),c((5,14))) = (1,0,0)$.

Observe that
$(c((2,13)), c((4,13)), c((6,13))) = (0,1,0) = (c((2,1)), c((4,1)), c((6,1)))$ and
$(c((1,14)), c((3,14)), c((5,14))) = (1,0,0) = (c((1,2)), c((3,2)), c((5,2)))$.
This implies that
$(c((2,j)), c((4,j)), c((6,j))) = (c((2,j+12)), c((4,j+12)), c((6,j+12)))$ for odd $j \geq 3$
and
$(c((1,j)), c((3,j)), c((5,j))) = (c((1,j+12)), c((3,j+12)), c((5,j+12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 2.2. $c((3,2)) = 1$ and $c((4,1)) = 0$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 1$. Hence,
$(c((2,1)), c((4,1)), c((6,1))) = (0,0,0)$ and $(c((1,2)), c((3,2)), c((5,2))) = (1,1,1)$.
$(\sigma((2,2)), \sigma((4,2)), \sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)), c((4,3)), c((6,3))) = (1,1,1)$.
$(\sigma((1,3)), \sigma((3,3)), \sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)), c((3,4)), c((5,4))) = (0,0,0)$.
$(\sigma((2,4)), \sigma((4,4)), \sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)), c((4,5)), c((6,5))) = (0,0,0)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
$(\sigma((1,5)), \sigma((3,5)), \sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)), c((3,6)), c((5,6))) = (1,1,1)$.

Observe that
$(c((2,5)), c((4,5)), c((6,5))) = (0,0,0) = (c((2,1)), c((4,1)), c((6,1)))$ and
$(c((1,6)), c((3,6)), c((5,6))) = (1,1,1) = (c((1,2)), c((3,2)), c((5,2)))$.
This implies that
$(c((2,j)), c((4,j)), c((6,j))) = (c((2,j+4)), c((4,j+4)), c((6,j+4)))$ for odd $j \geq 3$
and
$(c((1,j)), c((3,j)), c((5,j))) = (c((1,j+4)), c((3,j+4), c((5,j+4)))$ for even $j \geq 4$.

As we have obtained a contradiction for $n = 5$, we have a contradiction for $n = 9$, and so on.

Case 3. $c((2,1)) = 1$ and $c((1,2)) = c((6,1)) = 0$.

$\sigma((3,1)) = 1$ implies that $c((3,2)) = c((4,1))$.

Subcase 3.1. $c((3,2)) = c((4,1)) = 0$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 1$. Hence,
$(c((2,1)), c((4,1)), c((6,1))) = (1,0,0)$ and $(c((1,2)), c((3,2)), c((5,2))) = (0,0,1)$.
$(\sigma((2,2)), \sigma((4,2)), \sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)), c((4,3)), c((6,3))) = (0,0,0)$.
$(\sigma((1,3)), \sigma((3,3)), \sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)), c((3,4)), c((5,4))) = (1,1,0)$.
$(\sigma((2,4)), \sigma((4,4)), \sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)), c((4,5)), c((6,5))) = (1,0,0)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
$(\sigma((1,5)), \sigma((3,5)), \sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)), c((3,6)), c((5,6))) = (1,1,1)$.
$(\sigma((2,6)), \sigma((4,6)), \sigma((6,6))) = (1,1,1) \Rightarrow (c((2,7)), c((4,7)), c((6,7))) = (0,1,1)$.
$(\sigma((1,7)), \sigma((3,7)), \sigma((5,7))) = (1,1,1) \Rightarrow (c((1,8)), c((3,8)), c((5,8))) = (1,1,0)$.
$(\sigma((2,8)), \sigma((4,8)), \sigma((6,8))) = (1,1,1) \Rightarrow (c((2,9)), c((4,9)), c((6,9))) = (1,1,1)$.

If $n = 9$, then $\sigma((5,9)) = 0$, a contradiction. Hence assume that $n \geq 13$.

$(\sigma((1,9)),\sigma((3,9)),\sigma((5,9))) = (1,1,1) \Rightarrow (c((1,10)),c((3,10)),c((5,10))) = (0,0,1)$.
$(\sigma((2,10)),\sigma((4,10)),\sigma((6,10))) = (1,1,1) \Rightarrow (c((2,11)),c((4,11)),c((6,11))) = (0,1,1)$.
$(\sigma((1,11)),\sigma((3,11)),\sigma((5,11))) = (1,1,1) \Rightarrow (c((1,12)),c((3,12)),c((5,12))) = (0,0,0)$.
$(\sigma((2,12)),\sigma((4,12)),\sigma((6,12))) = (1,1,1) \Rightarrow (c((2,13)),c((4,13)),c((6,13))) = (1,0,0)$.

If $n = 13$, then $\sigma((1,13)) = 0$, a contradiction. Hence assume that $n \geq 17$.
$(\sigma((1,13)),\sigma((3,13)),\sigma((5,13))) = (1,1,1) \Rightarrow (c((1,14)),c((3,14)),c((5,14))) = (0,0,1)$.

Observe that
$(c((2,13)),c((4,13)),c((6,13))) = (1,0,0) = (c((2,1)),c((4,1)),c((6,1)))$ and
$(c((1,14)),c((3,14)),c((5,14))) = (0,0,1) = (c((1,2)),c((3,2)),c((5,2)))$.
This implies that
$(c((2,j)),c((4,j)),c((6,j))) = (c((2,j+12)),c((4,j+12)),c((6,j+12)))$ for odd $j \geq 3$
and
$(c((1,j)),c((3,j)),c((5,j))) = (c((1,j+12)),c((3,j+12)),c((5,j+12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 3.2. $c((3,2)) = c((4,1)) = 1$.

$\sigma((5,1)) = 1 \Rightarrow c((5,2)) = 0$. Hence,
$(c((2,1)),c((4,1)),c((6,1))) = (1,1,0)$ and $(c((1,2)),c((3,2)),c((5,2))) = (0,1,0)$.
$(\sigma((2,2)),\sigma((4,2)),\sigma((6,2))) = (1,1,1) \Rightarrow (c((2,3)),c((4,3)),c((6,3))) = (1,1,1)$.
$(\sigma((1,3)),\sigma((3,3)),\sigma((5,3))) = (1,1,1) \Rightarrow (c((1,4)),c((3,4)),c((5,4))) = (1,0,1)$.
$(\sigma((2,4)),\sigma((4,4)),\sigma((6,4))) = (1,1,1) \Rightarrow (c((2,5)),c((4,5)),c((6,5))) = (1,1,0)$.

If $n = 5$, then $\sigma((1,5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
$(\sigma((1,5)),\sigma((3,5)),\sigma((5,5))) = (1,1,1) \Rightarrow (c((1,6)),c((3,6)),c((5,6))) = (1,1,1)$.
$(\sigma((2,6)),\sigma((4,6)),\sigma((6,6))) = (1,1,1) \Rightarrow (c((2,7)),c((4,7)),c((6,7))) = (0,0,1)$.
$(\sigma((1,7)),\sigma((3,7)),\sigma((5,7))) = (1,1,1) \Rightarrow (c((1,8)),c((3,8)),c((5,8))) = (1,0,1)$.
$(\sigma((2,8)),\sigma((4,8)),\sigma((6,8))) = (1,1,1) \Rightarrow (c((2,9)),c((4,9)),c((6,9))) = (0,0,0)$.

If $n = 9$, then $\sigma((3,9)) = 0$, a contradiction. Hence assume that $n \geq 13$.
$(\sigma((1,9)),\sigma((3,9)),\sigma((5,9))) = (1,1,1) \Rightarrow (c((1,10)),c((3,10)),c((5,10))) = (0,1,0)$.
$(\sigma((2,10)),\sigma((4,10)),\sigma((6,10))) = (1,1,1) \Rightarrow (c((2,11)),c((4,11)),c((6,11))) = (0,0,1)$.
$(\sigma((1,11)),\sigma((3,11)),\sigma((5,11))) = (1,1,1) \Rightarrow (c((1,12)),c((3,12)),c((5,12))) = (0,0,0)$.
$(\sigma((2,12)),\sigma((4,12)),\sigma((6,12))) = (1,1,1) \Rightarrow (c((2,13)),c((4,13)),c((6,13))) = (1,1,0)$.

If $n = 13$, then $\sigma((3,13)) = 0$, a contradiction. Hence assume that $n \geq 17$.
$(\sigma((1,13)),\sigma((3,13)),\sigma((5,13))) = (1,1,1) \Rightarrow (c((1,14)),c((3,14)),c((5,14))) = (0,1,0)$.

Observe that
$(c((2,13)),c((4,13)),c((6,13))) = (1,1,0) = (c((2,1)),c((4,1)),c((6,1)))$ and
$(c((1,14)),c((3,14)),c((5,14))) = (0,1,0) = (c((1,2)),c((3,2)),c((5,2)))$.
This implies that
$(c((2,j)),c((4,j)),c((6,j))) = (c((2,j+12)),c((4,j+12)),c((6,j+12)))$ for odd $j \geq 3$
and

$(c((1,j)), c((3,j)), c((5,j))) = (c((1,j+12)), c((3,j+12)), c((5,j+12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

This completes the proof.

□

3. Conclusion

For the left over cases, we conjecture that: (i) if $n \in \{14, 26, 38, \ldots, 12r+2, \ldots\} \cup \{16, 28, 40, \ldots, 12r+4, \ldots\} \cup \{8, 20, 32, \ldots, 12r+8, \ldots\} \cup \{22, 34, \ldots, 12r+10, \ldots\}$, then $mc(C_3 \Box P_n) = 3$, and (ii) if $m \equiv 2 \mod 4$, $m \geq 10$, and $n \equiv 1 \mod 4$, $n \geq 5$, then $mc(C_m \Box P_n) = 3$.

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N. Paramaguru

Mathematics Wing, Directorate of Distance Education, Annamalai University, Annamalainagar-608 002, India

Email: npguru@gmail.com

R. Sampathkumar

Mathematics Section, Faculty of Engineering and Technology, Annamalai University, Annamalainagar-608 002, India

Email: sampathmath@gmail.com