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THE LAPLACIAN AND DISTANCE MATRIX OF A SIGNED TREE

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Abstract. Let N and $\widetilde{D}$ be net Laplacian and net distance matrices of a signed tree, respectively. The inverse (resp. group inverse) of $\widetilde{D}$ is obtained if it is nonsingular (resp. singular), which extend the inverse formula obtained by Graham and Lovász for distance matrix of a unsigned tree. The interlacing inequality connecting the eigenvalues of $\widetilde{D}$ and N of a signed tree is also obtained.

1. Introduction

A signed graph $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple graph called the underlying graph, and $\sigma : E \to \{1, -1\}$ is the signature. An edge $e \in E$ of $\dot{G}$ is called positive (resp. negative) if $\sigma(e) = +1$ (resp. $\sigma(e) = -1$). We use $|E^+_{\dot{G}}|$ (resp. $|E^-_{\dot{G}}|$) to denote the number of positive (resp. negative) edges of $\dot{G}$. The number of vertices of $\dot{G}$ is denoted by n. We use $i \sim j$ to denote that vertex i is adjacent to vertex j. While $i \nsim j$ means that vertex i is not adjacent to vertex j.

For a signed graph $\dot{G} = (G, \sigma)$, the degree δ_i of a vertex i of $\dot{G}$ is the number of its neighbours. The positive degree (resp. negative degree) δ^+_i (resp. δ^-_i) is the number of positive (resp. negative) neighbours of i. The net degree of i is defined to be $\delta^{\pm}_i = \delta^+_i - \delta^-_i$. The adjacency matrix $A = (a_{ij})$ of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The net Laplacian matrix is defined as $N = \triangle^{\pm} - A$, where $\triangle^{\pm}$ is the diagonal matrix of vertex net degrees. For basic properties of the net Laplacian matrix, we can refer

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[11, 12]. The Laplacian matrix is $L = \triangle - A$, where $\triangle$ denotes the diagonal matrix of vertex degrees. One can see [9] for basic properties of the Laplacian matrix of unsigned graphs.

Recall that the distance $d(i,j)$ between the vertices i and j of G is the length of a shortest path from i to j. The distance matrix $D = (d_{ij})$ of G is an $n \times n$ matrix, where $d_{ij} = d(i,j)$ and $d_{ii} = 0$, $i,j = 1,2,\ldots,n$. We can refer [3] for basic properties of distance matrix. Let $\dot{T}$ be a signed tree with n vertices and we recall that any two vertices i and j are joined by a unique path $\mathcal{P}_{i,j}$. In a signed tree $\dot{T}$, for two vertices i and j, $\tilde{d}(i,j) = \sum_{e\in\mathcal{P}_{i,j}} \sigma(e)$ denotes the net distance between i and j, specially, $\tilde{d}(i,i) = 0,\ i = 1,2,\ldots,n$. The net distance matrix of a signed tree $\dot{T}$ is defined as $\widetilde{D} = (\tilde{d}_{ij})$, where $\tilde{d}_{ij} = \tilde{d}(i,j), 1 \le i,j \le n$. Recall that for an $n \times n$ matrix M, the group inverse of M, when it exist, is the unique $n \times n$ matrix X satisfying the matrix equations $(i) MXM = M, (ii) XMX = X, (iii) MX = XM$. It is customary to denote the group inverse of M by $M^{\#}$.

Graham and Pollak [7] have shown that if T is a tree on n vertices with distance matrix D, then the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$, which depends only on the order of T, independent of the structure of T. They also take into account the inertia of D which is a triple of integers $(n_+(D), n_0(D), n_-(D))$, where $n_+(D)$, $n_0(D)$ and $n_-(D)$ denote the number of positive eigenvalues of D, the multiplicity of 0 as the eigenvalue of D and the number of negative eigenvalues of D, respectively. [6] obtained a formula for D^{-1}, that is, $D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau^T$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T, \tau_i = 2 - \delta_i,\ i = 1,2,\ldots,n$. See [2] also. The formula establishes the relationship between the inverse of the distance matrix and Laplacian matrix of a tree. Merris [10] gave the interlacing inequalities connecting the eigenvalues of distance and Laplacian matrices of a tree. Atik et al. [1] investigated some properties of the distance matrices of matrix weighted trees in connection with the Laplacian matrices, incidence matrices, and g-inverses, and they derived an interlacing inequality for the eigenvalues of distance and Laplacian matrices for the case of positive definite matrix weights. More related about distance matrix D of a tree, see [2, 4, 8, 13].

In this paper, we generalize the above results to a signed tree. Different from the results above, the weight of a edge may be a negative number. The paper is organized as follows. In Section 2, we obtain the determinant of the net distance matrix $\widetilde{D}$ of a signed tree and $\widetilde{D}$ is nonsingular if and only if $|E^+_{\dot{T}}| \neq |E^-_{\dot{T}}|$. Furthermore, if $\widetilde{D}$ is nonsingular, then $\widetilde{D}^{-1} = -\frac{1}{2}N + \frac{1}{2(p-q)}\tau\tau^T$, where $|E^+_{\dot{T}}| = p$, $|E^-_{\dot{T}}| = q$ and τ is a $n \times 1$ vector with $\tau_i = 2 - \delta_i,\ i = 1,2,\ldots,n$. In Section 3, we obtain the explicit formulas for $N^{\#}$ and $\widetilde{D}^{\#}$, that are, $N^{\#} = -\frac{1}{2}\widetilde{D} - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J + \frac{1}{2n}(\widetilde{D}J + J\widetilde{D})$ and $\widetilde{D}^{\#} = -\frac{1}{2}N - \frac{\tau^T N\tau}{2(\tau^T\tau)^2}\tau\tau^T + \frac{1}{2\tau^T\tau}(N\tau\tau^T + \tau\tau^T N)$. In Section 4, we drive a interlacing inequality connecting the eigenvalues of $\widetilde{D}$ and N of a signed tree.

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2. The formula for $\widetilde{D}^{-1}$

In what follows we assume that $|E^+_{\dot{T}}| = p$ and $|E^-_{\dot{T}}| = q$ for a signed tree $\dot{T}$. We know that there is a unique path between any two vertices of $\dot{T}$. Let $\theta(1) = 0$ and $\theta(i)$ $(i \geq 2)$ be the sign of the last edge of the path $\mathcal{P}_{1,i}$. Define the $n \times n$ matrix B by

$$B = \begin{pmatrix} 0 & \theta(2) & \theta(3) & \cdots & \theta(n) \\ \theta(2) & -2\theta(2) & 0 & \cdots & 0 \\ \theta(3) & 0 & -2\theta(3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta(n) & 0 & 0 & \cdots & -2\theta(n) \end{pmatrix}.$$

Let $P = (p_{ij})$ be an $n \times n$ matrix such that $p_{ij} = 1$ if $d_{1j} = d_{1i} + d_{ij}$, and $p_{ij} = 0$, otherwise. Thus, $p_{ij} = 1$ if and only if the unique path from 1 to j contains i. It is obvious that for a signed tree, P is a nonsingular matrix and $det(P) = 1$.

In this section, following the method of [6] for finding the inverse of distance matrix for a unsigned tree, we first show that $\widetilde{D} = P^T BP$ and also obtain the determinant of $\widetilde{D}$. Furthermore, we get the inertia of $\widetilde{D}$. At last we have the formula for $\widetilde{D}^{-1}$ when it is nonsingular.

Lemma 2.1. *Let $\dot{T}$ be a signed tree of order n with net distance matrix $\widetilde{D}$. Then*

$$\widetilde{D} = P^T BP.$$

Proof. We prove this identity by stating that the entries on the left and right are equal. Let $BP = (c_{ij})$, $B = (b_{ij})$. Then we obtain $c_{1j} = \sum_{k=1}^n b_{1k}p_{kj} = \sum_{e\in\mathcal{P}_{1,j}} \sigma(e) = \tilde{d}_{1j}$, while for $i > 1$, $c_{ij} = \sum_{k=1}^n b_{ik}p_{kj} = \theta(i) - 2\theta(i)p_{ij}$. Writing $P^T BP = (c'_{ij})$, we get

$$\begin{aligned} c'_{ij} = \sum_{k=1}^{n} p_{ki}c_{kj} &= p_{1i}c_{1j} + \sum_{k=2}^{n} p_{ki}(\theta(k) - 2\theta(k))p_{kj} \\ &= \tilde{d}_{1j} + \sum_{k=2}^{n} p_{ki}\theta(k) - 2\sum_{k=2}^{n} p_{ki}p_{kj}\theta(k) \\ &= \tilde{d}_{1j} + \tilde{d}_{1i} - 2\sum_{e\in\mathcal{P}_{1,i}\cap\mathcal{P}_{1,j}} \sigma(e) = \tilde{d}_{ij}. \end{aligned}$$

□

Remark 2.2. *According to the above lemma, we have $det(\widetilde{D}) = det(B) = (-1)^p(p-q)2^{n-2}$, which depends only on the number of positive and negative edges, and not on the structure of the signed tree. The inertia of $\widetilde{D}$ is the same as B. $\widetilde{D}$ is nonsingular if and only if $p \neq q$. Then, for a signed tree with*

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p positive edges and q negative edges, the inertia of $\widetilde{D}$ is given by

$$(n_+(\widetilde{D}), n_0(\widetilde{D}), n_-(\widetilde{D})) = \begin{cases} (q+1, 0, p), & p > q, \\ (q, 1, p), & p = q, \\ (q, 0, p+1), & p < q. \end{cases}$$

The following lemma gives the inverse of matrix P.

Lemma 2.3. [6, Fact 2] *The inverse of P, denoted by $P^{-1} = (p^*_{ij})$, is the $n \times n$ matrix defined as follows.*

$$p^*_{ij} = \begin{cases} 1, & if\ i = j, \\ -1, & if\ i \sim j\ and\ \tilde{d}_{1j} = \sigma(e_{ij}) + \tilde{d}_{1i}, \\ 0, & otherwise. \end{cases}$$

The following lemma gives a connection between the diagonal matrix $diag(0, \theta(2), \ldots, \theta(n))$, N and P, which will be used in the sequel.

Lemma 2.4. *Let $\dot{T}$ be a signed tree on n vertices with net Laplacian matrix $N = (n_{ij})$. Then*

$$N = P^{-1} \begin{pmatrix} 0 & & & \\ & \theta(2) & & \\ & & \ddots & \\ & & & \theta(n) \end{pmatrix} (P^{-1})^T.$$

Proof. We prove the lemma by stating that the entries on the left and right are equal. Let

$$Pdiag(0, \theta(2), \ldots, \theta(n))(P^{-1})^T = X = (x_{ij}).$$

Then according to the definition of matrix multiplication we have $x_{ij} = \sum_{k=1}^{n} p^*_{ik}\theta(k)p^*_{jk}$. If $i = j$, we obtain $x_{ii} = \sum_{k=1}^{n}(p^*_{ik})^2\theta(k) = \sum_{k \sim i}\sigma(ki) = \delta^{\pm}_i = n_{ii}$. If $i \neq j$, we cannot have $p^*_{ik} = p^*_{jk} \neq 0$. Thus, the only nonzero contribution can occur if $p^*_{ik} = -p^*_{jk}$, $i \sim j$, then $x_{ij} = -\sigma(ij) = n_{ij}$. □

Remark 2.5. *For a signed tree with p positive edges and q negative edges, by Lemma 2.4, we have the inertia of N is $(p, 1, q)$, which was obtained in* [5].

Let τ be an $n \times 1$ vector defined by $\tau = (2 - \delta_1, 2 - \delta_2, \ldots, 2 - \delta_n)^T$.

Lemma 2.6. *Let $\dot{T}$ be a signed tree with n vertices and $\xi = (2, 1, \ldots, 1)^T$. Then*

$$P^{-1}\xi\xi^T(P^{-1})^T = \tau\tau^T.$$

Proof. We prove the lemma by stating that the entries on the left and right are equal. The (i, j) entry of $P^{-1}\xi\xi^T(P^{-1})^T$ is $(2p^*_{i1} + p^*_{i2} + \cdots + p^*_{in})(2p^*_{j1} + p^*_{j2} + \cdots + p^*_{jn})$ which is $(2 - \delta_i)^2$, if $i = j = 1$, and $(2 - \delta_i)(2 - \delta_j)$, otherwise. It is corresponding to the entry of $\tau\tau^T$. □

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For a signed tree $\dot{T}$ with $p \neq q$, the corresponding matrix B is nonsingular and its inverse is

$$B^{-1} = \frac{1}{2(p-q)} \begin{pmatrix} 4 & 2 & 2 & \cdots & 2 \\ 2 & -(p-q)\theta(2)+1 & 1 & \cdots & 1 \\ 2 & 1 & -(p-q)\theta(3)+1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 1 & \cdots & -(p-q)\theta(n)+1 \end{pmatrix}.$$

Using $\widetilde{D}^{-1} = P^{-1}B^{-1}(P^{-1})^T$ and the previous lemmas, we obtain the following formula for $\widetilde{D}^{-1}$, which appeared in [8] in term of weighted form.

Theorem 2.7. *Let $\dot{T}$ be a signed tree of order n with $p \neq q$. Let N and $\widetilde{D}$ be net Laplacian matrix and net distance matrix of $\dot{T}$, respectively. Then*

$$\widetilde{D}^{-1} = -\frac{1}{2}N + \frac{1}{2(p-q)}\tau\tau^T.$$

Proof. For $p \neq q$, the matrices B and $\widetilde{D}$ are nonsingular. By Lemma **??**, it follows that

$$\begin{aligned}
\widetilde{D}^{-1} &= P^{-1}B^{-1}(P^{-1})^T \\
&= \frac{1}{2(p-q)}P^{-1} \begin{pmatrix} 4 & 2 & 2 & \cdots & 2 \\ 2 & -(p-q)\theta(2)+1 & 1 & \cdots & 1 \\ 2 & 1 & -(p-q)\theta(3)+1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 1 & \cdots & -(p-q)\theta(n)+1 \end{pmatrix} (P^{-1})^T \\
&= \frac{1}{2(p-q)}P^{-1} \begin{pmatrix} 0 & & & \\ & -(p-q)\theta(2) & & \\ & & \ddots & \\ & & & -(p-q)\theta(n) \end{pmatrix} (P^{-1})^T \\
&+ \frac{1}{2(p-q)}P^{-1} \begin{pmatrix} 4 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & \cdots & 1 \end{pmatrix} (P^{-1})^T \\
&= -\frac{1}{2}P^{-1} \begin{pmatrix} 0 & & & \\ & \theta(2) & & \\ & & \ddots & \\ & & & \theta(n) \end{pmatrix} (P^{-1})^T + \frac{1}{2(p-q)}P^{-1} \begin{pmatrix} 2 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & \cdots & 1 \end{pmatrix} (P^{-1})^T \\
&= -\frac{1}{2}N + \frac{1}{2(p-q)}\tau\tau^T.
\end{aligned}$$

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□

Example 2.8. *We consider a signed tree (see Figure 1) of order 6 with $p = 3,\ q = 2$. In figures of signed graphs below, dashed lines represent the negative edges and solid lines represent the positive edges.*

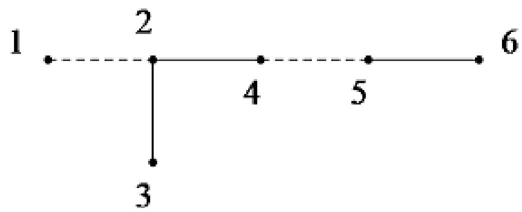

FIGURE 1. A signed tree of order six with $p = 3,\ q = 2$

The net distance matrix and the net Laplacian matrix are given by

$$\widetilde{D} = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{pmatrix},\ N = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

and $\tau = (1, -1, 1, 0, 0, 1)^T$. Then

$$\widetilde{D}^{-1} = \begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} = -\frac{1}{2}N + \frac{1}{2(p-q)}\tau\tau^T.$$

3. Formulas for $N^{\#}$ and $\widetilde{D}^{\#}$

Let $\mathbf{1}$ be all ones vector and J be all ones matrix. Let $A[\{\bar{i}\}|\{\bar{i}\}]$ be the principle submatrix obtained from A by deleting its i-th row and i-th column. If i is a vertex of $\dot{T}$, then $\dot{T}\backslash\{i\}$ denotes the signed graph obtained from $\dot{T}$ by removing i. In this section, we first drive the eigenvector associated with eigenvalue 0 of $\widetilde{D}$ when $\widetilde{D}$ is singular. Furthermore, we obtain the group inverses of N and $\widetilde{D}$, respectively. The following proofs of Lemmas 3.1 and 3.2 are similar to those of unsigned trees [2, 3, 8]. For the completeness, we provide the proofs here.

Lemma 3.1. *Let $\dot{T}$ be a signed tree with p positive edges and q negative edges, and $\widetilde{D}$ be the net distance matrix of $\dot{T}$. Then*

$$\widetilde{D}\tau = (p-q)\mathbf{1}.$$

Proof. We prove the result by induction on the number of vertices $n = p+q+1$ of $\dot{T}$. When $n=1$ the result is clear. For $n=2$, if $\widetilde{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $p=1$, $q=0$, $\tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we get $\widetilde{D}\tau = \mathbf{1}$. When $\widetilde{D} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, then $p=0$, $q=1$, $\tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we get $\widetilde{D}\tau = -\mathbf{1}$. So let $n \geq 3$ and hypothesize the conclusion is true for signed trees with less than n vertices. Next we consider a signed tree with n vertices. Without losing generality, we assume that the vertex n is pendant and adjacent to vertex $n-1$ in $\dot{T}$. Thus, $\tau_n = 2 - d_n = 1$. Partition $\widetilde{D}$ and τ as

$$\widetilde{D} = \begin{pmatrix} \widetilde{D}[\bar{\{n\}}|\bar{\{n\}}] & \alpha \\ \alpha^T & 0 \end{pmatrix}, \ \tau = \begin{pmatrix} \hat{\tau} \\ 1 \end{pmatrix}.$$

Then

$$\widetilde{D}\tau = \begin{pmatrix} \widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]\hat{\tau} + \alpha \\ \alpha^T\hat{\tau} \end{pmatrix}.$$

The net distance matrix of $\dot{T}\backslash\{n\}$ is $\widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]$. While the degree of vertex $n-1$ in $\dot{T}\backslash\{n\}$ is $d_{n-1}-1$. Let $\tilde{\tau} = \hat{\tau} + (0,0,\ldots,1)^T$. According to the inductive hypothesis, we get $\widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]\tilde{\tau} = (p-q-\sigma(n-1,n))\mathbf{1}$. Let β be the last column of $\widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]$, then

$$\widetilde{D}[\bar{\{n\}}|\bar{\{n\}}](\hat{\tau} + (0,0,\ldots,1)^T) = \widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]\hat{\tau} + \beta.$$

It follows that

$$\widetilde{D}(\bar{\{n\}}|\bar{\{n\}})\hat{\tau} = (p-q-\sigma(n-1,n))\mathbf{1} - \beta.$$

Since $\tilde{d}_{in} = \tilde{d}_{i,n-1} + \sigma(n-1,n), \ i = 1,2,\ldots,n-1$, then $\alpha = \beta + \sigma(n-1,n)\mathbf{1}$. It follows that

$$\widetilde{D}[\bar{\{n\}}|\bar{\{n\}}]\hat{\tau} + \alpha = (p-q-\sigma(n-1,n))\mathbf{1} - \beta + \beta + \sigma(n-1,n)\mathbf{1} = (p-q)\mathbf{1}.$$

Since a signed tree on n ($n \geq 3$) vertices has at least two pendant vertices, it is possible to repeat the above argument for another pendant vertex. Thus the equation holds for a signed tree with n vertices. □

According to the above lemma, we know that if $\widetilde{D}$ is singular then τ is an eigenvector associated with eigenvalue 0.

Lemma 3.2. *Let $\widetilde{D}$ and N be the net distance matrix and the net Laplacian matrix of $\dot{T}$ with n vertices, respectively. Then*

$$N\tilde{D} + 2I = \tau\mathbf{1}^T.$$

Proof. For vertices $i,\ j \in \{1, 2, \dots, n\}$, we suppose that the degree of vertex i is ℓ, and without loss generality, let vertex i be adjacent to $1, 2, \dots, \ell$. If $i \neq j$, the graph $\dot{T}\backslash\{i\}$ is a forest with ℓ components. Without losing generality, we assume that vertex j is in the component of $\dot{T}\backslash\{i\}$ containing vertex 1. Then, we have $\tilde{d}_{\upsilon j} = \tilde{d}_{1j} + \sigma(1i) + \sigma(\upsilon i) = \tilde{d}_{ij} + \sigma(\upsilon i)$, $\upsilon = 2, 3, \dots, \ell$. So

$$\begin{aligned}
(N\widetilde{D} + 2I)_{ij} = (N\widetilde{D})_{ij} &= \sum_{k=1}^{n} n_{ik}\tilde{d}_{kj} = \delta_i^{\pm}\tilde{d}_{ij} - \sum_{\upsilon=1}^{\ell} \sigma(i\upsilon)\tilde{d}_{\upsilon j} \\
&= \delta_i^{\pm}\tilde{d}_{ij} - \sum_{\upsilon=2}^{\ell} \sigma(i\upsilon)(\tilde{d}_{ij} + \sigma(i\upsilon)) - \sigma(i1)\tilde{d}_{1j} \\
&= \delta_i^{\pm}\tilde{d}_{ij} - \sum_{\upsilon=2}^{\ell} \sigma(i\upsilon)\tilde{d}_{ij} - \sum_{\upsilon=2}^{\ell} 1 - \sigma(i1)\tilde{d}_{1j} \\
&= (\delta_i^{\pm} - \sum_{\upsilon=1}^{\ell} \sigma(i\upsilon))\tilde{d}_{ij} + \sigma(i1)\tilde{d}_{ij} - (\ell - 1) - \sigma(i1)\tilde{d}_{1j} \\
&= \sigma(i1)(\tilde{d}_{ij} - \tilde{d}_{1j}) - \ell + 1 \\
&= \sigma(i1)^2 - \ell + 1 = 2 - \ell = \tau_i.
\end{aligned}$$

When $i = j$ we have

$$\begin{aligned}
(N\widetilde{D} + 2I)_{ii} &= \delta_i^{\pm}\tilde{d}_{ii} - (\sigma(i1)\tilde{d}_{1i} + \cdots + \sigma(i\ell)\tilde{d}_{\ell i}) + 2 \\
&= -[\sigma(i1)^2 + \cdots + \sigma(i\ell)^2] + 2 = 2 - \ell = \tau_i.
\end{aligned}$$

In conclusion, for any $i,\ j$, we have $(N\widetilde{D} + 2I)_{ij} = \tau_i$. Therefore, $N\widetilde{D} + 2I = \tau\mathbf{1}^T$. □

For a signed tree, recall Remark 2.5, $rank(N) = n - 1$. The following theorem gives a formula for $N^{\#}$, which can be represented as the sum of $-\frac{1}{2}\widetilde{D}$ and two matrices with rank 1.

Theorem 3.3. *Let $\widetilde{D}$ and N be the net distance matrix and the net Laplacian matrix of a signed tree $\dot{T}$ with n vertices, respectively. Then*

$$N^{\#} = -\frac{1}{2}\widetilde{D} - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J + \frac{1}{2n}(\widetilde{D}J + J\widetilde{D}).$$

Proof. Let $H = -\frac{1}{2}\widetilde{D} - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J + \frac{1}{2n}(\widetilde{D}J + J\widetilde{D})$. We prove the theorem by the definition of group inverse. In fact

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$$\begin{aligned}
NH &= -\frac{1}{2}N\widetilde{D} - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}NJ + \frac{1}{2n}(N\widetilde{D}J + NJ\widetilde{D}) \\
&= -\frac{1}{2}(\tau\mathbf{1}^T - 2I) + \frac{1}{2n}(\tau\mathbf{1}^T - 2I)J \\
&= -\frac{1}{2}\tau\mathbf{1}^T + I + \frac{1}{2}\tau\mathbf{1}^T - \frac{1}{n}J \\
&= I - \frac{1}{n}J,
\end{aligned}$$

and

$$\begin{aligned}
HN &= -\frac{1}{2}\widetilde{D}N - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}JN + \frac{1}{2n}(\widetilde{D}JN + J\widetilde{D}N) \\
&= -\frac{1}{2}(\mathbf{1}\tau^T - 2I) + \frac{1}{2n}J(\mathbf{1}\tau^T - 2I) \\
&= -\frac{1}{2}\mathbf{1}\tau^T + I + \frac{1}{2}\mathbf{1}\tau^T - \frac{1}{n}J \\
&= I - \frac{1}{n}J,
\end{aligned}$$

hence

$$NH = HN.$$

While

$$NHN = (I - \frac{1}{n}J)N = N.$$

And we have

$$\begin{aligned}
HNH &= (I - \frac{1}{n}J)(-\frac{1}{2}\widetilde{D} - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J + \frac{1}{2n}(\widetilde{D}J + J\widetilde{D})) \\
&= H + \frac{1}{2n}J\widetilde{D} + \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^3}JJ - \frac{1}{2n^2}J\widetilde{D}J - \frac{1}{2n^2}JJ\widetilde{D} \\
&= H + \frac{1}{2n}J\widetilde{D} + \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J - \frac{\mathbf{1}^T\widetilde{D}\mathbf{1}}{2n^2}J - \frac{1}{2n}J\widetilde{D} \\
&= H.
\end{aligned}$$

Thus H is the group inverse of N. □

For a signed tree with the same number of positive edges and negative edges, $\widetilde{D}$ is singular. The following theorem give the explicit formula of group inverse $\widetilde{D}^{\#}$ of $\widetilde{D}$, which extend to the inverse $\widetilde{D}^{-1}$ if $\widetilde{D}$ is nonsingular.

Theorem 3.4. *Let $\dot{T}$ be a signed tree on n vertices with net distance matrix $\widetilde{D}$ and net Laplacian matrix N. Suppose $|E_{\dot{T}}^{+}| = p$, $|E_{\dot{T}}^{-}| = q$ and $p = q$. Let $\widetilde{D}^{\#}$ be the group inverse of $\widetilde{D}$. Then*

$$\widetilde{D}^{\#} = -\frac{1}{2}N - \frac{\tau^T N\tau}{2(\tau^T\tau)^2}\tau\tau^T + \frac{1}{2\tau^T\tau}(N\tau\tau^T + \tau\tau^T N).$$

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Proof. Let $H=-\frac{1}{2}N-\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\tau\tau^T+\frac{1}{2\tau^T\tau}(N\tau\tau^T+\tau\tau^TN)$. We prove that H is the group inverse of $\widetilde{D}$ by the definition of group inverse. In fact

$$\begin{aligned}\widetilde{D}H&=-\frac{1}{2}\widetilde{D}N-\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\widetilde{D}\tau\tau^T+\frac{1}{2\tau^T\tau}(\widetilde{D}N\tau\tau^T+\widetilde{D}\tau\tau^TN)\\&=-\frac{1}{2}(\mathbf{1}\tau^T-2I)+\frac{1}{2\tau^T\tau}(\mathbf{1}\tau^T-2I)\tau\tau^T\\&=-\frac{1}{2}\mathbf{1}\tau^T+I+\frac{1}{2\tau^T\tau}\mathbf{1}\tau^T\tau\tau^T-\frac{1}{\tau^T\tau}\tau\tau^T\\&=I-\frac{1}{\tau^T\tau}\tau\tau^T,\end{aligned}$$

and

$$\begin{aligned}H\widetilde{D}&=-\frac{1}{2}N\widetilde{D}-\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\tau\tau^T\widetilde{D}+\frac{1}{2\tau^T\tau}(N\tau\tau^T\widetilde{D}+\tau\tau^TN\widetilde{D})\\&=-\frac{1}{2}(\tau\mathbf{1}^T-2I)+\frac{1}{2\tau^T\tau}\tau\tau^T(\tau\mathbf{1}^T-2I)\\&=-\frac{1}{2}\tau\mathbf{1}^T+I+\frac{1}{2\tau^T\tau}\tau\tau^T\tau\mathbf{1}^T-\frac{1}{\tau^T\tau}\tau\tau^T\\&=I-\frac{1}{\tau^T\tau}\tau\tau^T,\end{aligned}$$

thus

$$\widetilde{D}H=H\widetilde{D}.$$

Then

$$\widetilde{D}H\widetilde{D}=(I-\frac{1}{\tau^T\tau}\tau\tau^T)\widetilde{D}=\widetilde{D},$$

and

$$\begin{aligned}H\widetilde{D}H&=(I-\frac{1}{\tau^T\tau}\tau\tau^T)(-\frac{1}{2}N-\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\tau\tau^T+\frac{1}{2\tau^T\tau}(N\tau\tau^T+\tau\tau^TN))\\&=H+\frac{1}{2\tau^T\tau}\tau\tau^TN+\frac{\tau^TN\tau}{2(\tau^T\tau)^3}\tau\tau^T\tau\tau^T-\frac{1}{2(\tau^T\tau)^2}\tau\tau^TN\tau\tau^T-\frac{1}{2(\tau^T\tau)^2}\tau\tau^T\tau\tau^TN\\&=H+\frac{1}{2\tau^T\tau}\tau\tau^TN+\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\tau\tau^T-\frac{\tau^TN\tau}{2(\tau^T\tau)^2}\tau\tau^T-\frac{1}{2\tau^T\tau}\tau\tau^TN=H.\end{aligned}$$

Hence H is the group inverse of $\widetilde{D}$. □

Remark 3.5. *Since $\widetilde{D}$ is a symmetric matrix, the group inverse of $\widetilde{D}$ is the same as Moore-Penrose inverse of $\widetilde{D}$. Kurata and Bapat [8] showed the relation $\widetilde{D}^{\#}=-\frac{1}{2}N-\frac{\mathbf{1}^T\widetilde{D}^{\#}\mathbf{1}}{4}\tau\tau^T+\frac{1}{2}(\widetilde{D}^{\#}\mathbf{1}\tau^T+\tau\mathbf{1}^T\widetilde{D}^{\#})$, which is a relationship between $\widetilde{D}^{\#}$ and N. However, our result is an explicit formula for $\widetilde{D}^{\#}$ in term of N and τ.*

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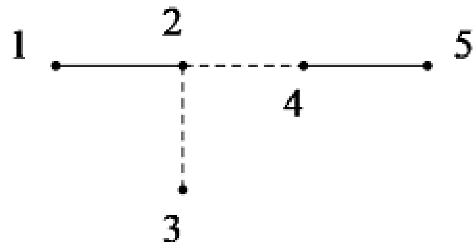

FIGURE 2. A signed tree of order five with $p = q = 2$

Example 3.6. *For a signed tree in Figure 2, we have* $\tau = (1, -1, 1, 0, 1)^T$, *and*

$$N = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \ \widetilde{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

We have

$$\widetilde{D}^{\#} = \begin{pmatrix} 0 & \frac{3}{8} & 0 & -\frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix} = -\frac{1}{2}N - \frac{\tau^T N \tau}{2(\tau^T\tau)^2}\tau\tau^T + \frac{1}{2\tau^T\tau}(N\tau\tau^T + \tau\tau^T N).$$

4. The eigenvalues of $\widetilde{D}$ and N

In this section we obtain the interlacing inequality connecting the eigenvalues of $\widetilde{D}$ and the eigenvalues of N of a signed tree, which extend the interlacing inequality connecting the eigenvalues of distance matrix and Laplacian matrix of a tree in [9]. Note that the interlacing inequality connecting the eigenvalues of the distance matrix and the Laplacian matrix of a weighted tree has been shown in [1, Theorem 3.8 and Corollary 3.9]. In these results, there exists a restriction that the weight is positive definite matrices or positive numbers. The theorem below consider the situation of that weight of edge may be negative, which provides a new sight to investigate the interlacing inequality.

Theorem 4.1. *Let $\dot{T}$ be a signed tree of order n with p positive edges and q negative edges. Let $\widetilde{D}$ and N be net distance matrix and net Laplacian matrix of $\dot{T}$, respectively, and the eigenvalues of $\widetilde{D}$ and N be arranged by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, respectively. Then*

$$\mu_1 \geq -\frac{2}{\lambda_{p+2}} \geq \mu_2 \geq \cdots \geq -\frac{2}{\lambda_n} \geq \mu_{n-p} \geq -\frac{2}{\lambda_1} \geq \mu_{n-p+1} \geq \cdots \geq -\frac{2}{\lambda_p} \geq \mu_n.$$

Proof. Recall that the inertia of N is $(p,1,q)$. Then the eigenvalues of $N^{\#}$ are $\frac{1}{\lambda_p} \geq \cdots \geq \frac{1}{\lambda_1} > 0 > \frac{1}{\lambda_n} \geq \cdots \geq \frac{1}{\lambda_{p+2}}$. Let Q be an orthogonal matrix such that

$$Q^T N^{\#} Q = diag(\frac{1}{\lambda_p}, \ldots, \frac{1}{\lambda_1}, 0, \frac{1}{\lambda_n}, \ldots, \frac{1}{\lambda_{p+2}}).$$

Then $(p+1)$th column of JQ is non-zero and remaining columns equal to zero. By Theorem 3.3, we have

$$\widetilde{D} = -2N^{\#} - \frac{\mathbf{1}^T \widetilde{D} \mathbf{1}}{n^2} J + \frac{1}{n}(\widetilde{D}J + J\widetilde{D}).$$

Hence $Q^T \widetilde{D} Q = diag(-\frac{2}{\lambda_p}, \ldots, -\frac{2}{\lambda_1}, 0, -\frac{2}{\lambda_n}, \ldots, -\frac{2}{\lambda_{p+2}}) - \frac{\mathbf{1}^T \widetilde{D} \mathbf{1}}{n^2} Q^T J Q + \frac{1}{n}(Q^T \widetilde{D} J Q + Q^T J \widetilde{D} Q)$. While matrix $-\frac{\mathbf{1}^T \widetilde{D} \mathbf{1}}{n^2} Q^T J Q + \frac{1}{n}(Q^T \widetilde{D} J Q + Q^T J \widetilde{D} Q)$ has entries of $(p+1)$th column, $(p+1)$th row non-zero and the others equal to zero. It follows that $diag(-\frac{2}{\lambda_p}, \ldots, -\frac{2}{\lambda_1}, -\frac{2}{\lambda_n}, \ldots, -\frac{2}{\lambda_{p+2}})$ is a principal submatrix of $Q^T \widetilde{D} Q$ of order $n-1$. By interlacing theorem, we have $\mu_1 \geq -\frac{2}{\lambda_{p+2}} \geq \mu_2 \geq \cdots \geq -\frac{2}{\lambda_n} \geq \mu_{n-p} \geq -\frac{2}{\lambda_1} \geq \mu_{n-p+1} \geq \cdots \geq -\frac{2}{\lambda_p} \geq \mu_n$. □

For the signed tree in Example 2.8, $p = 3$, $q = 2$. The eigenvalues of $\widetilde{D}$ are $3.862 \geq 1.613 \geq 0.478 \geq -0.872 \geq -2 \geq -3.018$ and the eigenvalues of N are $2.481 \geq 1.732 \geq 0.689 \geq 0 \geq -1.170 \geq -1.732$. We have $\mu_1 \geq -\frac{2}{\lambda_5} \geq \mu_2 \geq -\frac{2}{\lambda_6} \geq \mu_3 \geq -\frac{2}{\lambda_1} \geq \mu_4 \geq -\frac{2}{\lambda_2} \geq \mu_5 \geq -\frac{2}{\lambda_3} \geq \mu_6$.

For the signed tree in Example 3.6, $p = 2$, $q = 2$. The eigenvalues of $\widetilde{D}$ are $2.828 \geq 1.414 \geq 0 \geq -1.414 \geq -2.828$ and the eigenvalues of N are $1.872 \geq 1.370 \geq 0 \geq -0.797 \geq -2.446$. Thus we get $\mu_1 \geq -\frac{2}{\lambda_4} \geq \mu_2 \geq -\frac{2}{\lambda_5} \geq \mu_3 \geq -\frac{2}{\lambda_1} \geq \mu_4 \geq -\frac{2}{\lambda_2} \geq \mu_5$.

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