https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 15 No. 4 (2026), pp. 317–333.

https://www.ui.ac.ir

ANTIMAGIC LABELINGS ON GRAPHS WITH ASCENDING SUBGRAPH DECOMPOSITION

SIGIT PANCAHAYANI, RINOVIA SIMANJUNTAK* AND SALADIN UTTUNGGADEWA

Abstract. Let t and q be positive integers that satisfy $\binom{t+1}{2} \leq q < \binom{t+2}{2}$ and G be a simple and finite graph of size q. G is said to be an ascending subgraph decomposition (ASD) graph if G can be decomposed into t subgraphs $H_1, H_2, \ldots, H_t$ without isolated vertices such that H_i is isomorphic to a proper subgraph of H_{i+1}, for $1 \leq i \leq t-1$.

In this paper, we introduce a new type of antimagic labeling based on the notion of ASD. Let G be an ASD graph and $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ a bijection. The weight of a subgraph H_i ($1 \leq i \leq t$) is $w(H_i) = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$. If the weights of all H_is ($1 \leq i \leq t$) form an arithmetic progression with the smallest weight a and common difference d, then f is called an (a, d)-ASD antimagic labeling and G is an (a, d)-ASD antimagic graph.

We provide an upper bound for d in an (a, d)-ASD antimagic graph. We define and utilize the (t, δ)-ascending antibalanced multisets to label some product graphs, including disjoint union, vertex amalgamation, edge amalgamation, subgraph amalgamation, and extended chain of graphs.

1. Introduction

Let G be a simple and finite graph and $\mathcal{H} = \{H_i \mid i = 1, 2, 3, \ldots, t\}$ be a collection of t subgraphs of G such that each H_i is isomorphic to the same graph H. If each edge of G is contained in at least one

---

MSC(2010): Primary: 05C.

Keywords: ascending subgraph decomposition (ASD), antimagic labeling, (a, d)-ASD antimagic labeling.

Article Type: Research Paper.

Communicated by Ebrahim Ghorbani.

*Corresponding author.

Received: 31 October 2024, Accepted: 01 September 2025, Published Online: 11 January 2026.

Cite this article: S. Pancahayani, R. Simanjuntak and S. Uttunggadewa, Antimagic labelings on graphs with ascending subgraph decomposition, *Trans. Comb.*, **15** no. 4 (2026) 317-333. `https://dx.doi.org/10.22108/toc.2025.143242.2219` .

subgraph H_i, then G is said to admit an H-*covering* [10]. More specifically, if $\{E(H_1), \ldots, E(H_t)\}$ is a partition of $E(G)$ then G is said to admit an H-*decomposition* [5]. Note that both H-covering and H-decomposition of a graph require that all the subgraphs are isomorphic. However, in 1987, Alavi *et al.* [1] proposed a type of graph decomposition in which all subgraphs are not isomorphic to each other.

Definition 1. [1] *Let t and q be two positive integers that satisfy $\binom{t+1}{2} \leq q < \binom{t+2}{2}$ and G be a simple and finite graph of size q. G admits an* ***ascending subgraph decomposition*** *if G can be decomposed into t subgraphs $H_1, H_2, \ldots, H_t$ without isolated vertices such that H_i is isomorphic to a proper subgraph of H_{i+1} for $1 \leq i \leq t-1$. In this case, G is called an* ***ASD graph*** *and $H_1, H_2, \ldots, H_t$ are the* ***ascending subgraphs*** *of G.*

In the same article, Alavi *et. al.* proposed the following conjecture.

Conjecture 1. [1] *Every graph of positive size has an ascending subgraph decomposition.*

The conjecture remains open to this day, even though many families of graphs have been proven to be ASD (see the survey by Liang and Fu [9]).

We denote by $|V(G)|$ and $|E(G)|$ the order and size of G, respectively, and $\sigma(G) = |V(G)| + |E(G)|$. Let $f : V(G) \cup E(G) \to \{1, 2, \ldots, \sigma(G)\}$ be a bijection. We call the sum of labels of all vertices and edges of a subgraph H its weight, denoted by $w(H)$. If G admits an H-decomposition in which all the weights under f are distinct, then G is said to admit an H-antimagic total labeling [6]. Furthermore, if those weights form an arithmetic progression starting from a with difference d, then G admits an (a, d)-H-antimagic total labeling [6].

Combining the definitions of ascending subgraph decomposition and antimagic labeling leads us to define a new type of antimagic labeling of an ASD graph.

Definition 2. *Let G be an ASD graph with $\mathcal{H} = \{H_i \subseteq G \mid i = 1, 2, 3, \ldots, t\}$ a set of its ascending subgraphs and $f : V(G) \cup E(G) \to \{1, 2, \ldots, \sigma(G)\}$ be a bijection. If the weights of subgraphs in $\mathcal{H}$ form an arithmetic progression with the smallest value a and difference d, then G admits an (a, d)-ASD antimagic labeling.*

In this paper, we observe an upper bound for d in an (a, d)-ASD antimagic graph. We also define and utilize (t, δ)-ascending antibalanced multisets to label some product graphs, which include disjoint union, vertex amalgamation, edge amalgamation, subgraph amalgamation, and extended chain of graphs.

2. An Upper Bound

We start by observing an upper bound for the difference d in the (a, d)-ASD antimagic labeling on an ASD graph.

https://dx.doi.org/10.22108/toc.2025.143242.2219

Theorem 2.1. *For $t \geq 2$, let G be an ASD graph and $\mathcal{H} = \{H_i \mid i = 1, 2, \ldots, t\}$ be a collection of ascending subgraphs of G. If G is (a, d)-ASD antimagic, then*

$$d \leq \frac{\sigma(H_t)[2\sigma(G) - \sigma(H_t) + 1] - \sigma(H_1)[\sigma(H_1) + 1]}{2(t-1)}.$$

Proof. If G is (a, d)-ASD antimagic, then the largest possible weight is $a + (t-1)d = w(H_t)$, in which the labels used are $\sigma(G) - \sigma(H_t) + 1, \sigma(G) - \sigma(H_t) + 2, \ldots, \sigma(G) - 2, \sigma(G) - 1, \sigma(G)$. Therefore,

$$a + (t-1)d \leq \frac{\sigma(H_t)[2\sigma(G) - \sigma(H_t) + 1]}{2}.$$

On the other hand, the smallest weight is $a = w(H_1)$, in which the labels used are $1, 2, \ldots, \sigma(H_1)$. So,

$$a \geq \frac{\sigma(H_1)[\sigma(H_1) + 1]}{2}.$$

Therefore,

$$(t-1)d \leq \frac{\sigma(H_t)[2\sigma(G) - \sigma(H_t) + 1] - \sigma(H_1)[\sigma(H_1) + 1]}{2}.$$

Since $t \geq 2$,

$$d \leq \frac{\sigma(H_t)[2\sigma(G) - \sigma(H_t) + 1] - \sigma(H_1)[\sigma(H_1) + 1]}{2(t-1)}.$$

□

3. (t, δ)-Ascending Antibalanced Multisets

Recall that a multiset is a set that allows multiple identical elements. Let X and Y be multisets, then the union of X and Y, $X \uplus Y = \{\alpha \mid \alpha \in X; \alpha \in Y\}$ is also a multiset that lists all the elements in both X and Y. Moreover, define $\sum X = \sum_{x \in X} x$, as a summation of all elements of X.

Inayah *et al.* in [8] proposed a concept of (t, δ)-antibalanced as a tool to construct H-(a, d)-antimagic total labelings.

Definition 3. [8] *Let $t, \delta \in \mathbb{N}$. A multiset X is said to be a (t, δ)-**antibalanced multiset** if there exist t submultisets of X, say $X_1, X_2, X_3, \ldots, X_t$, such that for every $i \in [1, t]$, $|X_i| = \frac{|X|}{t}$, $\uplus_{i=1}^{t} X_i = X$, and for $i \in [1, t-1]$ satisfies $\sum X_{i+1} - \sum X_i = \delta$.*

In the previous definition, all submultisets have the same cardinality. In this paper, we propose an analogous definition where all submultisets have different cardinalities.

Definition 4. *Let $t, \delta \in \mathbb{N}$ and let X be a multiset consisting of positive integers. Then X is said to be a (t, δ)-**ascending antibalanced multiset** if there exist t submultisets of X, say $X_1, X_2, \ldots, X_t$ such that for every $i \in [1, t-1]$, $|X_i| < |X_{i+1}|$, $\sum_{i=1}^{t} |X_i| = |X|$, $\biguplus_{i=1}^{t} X_i = X$, and $\sum X_{i+1} - \sum X_i = \delta$.*

https://dx.doi.org/10.22108/toc.2025.143242.2219

Note that Definition 4 also may apply to sets that are not multisets. We construct some multisets that are (t,δ)-ascending antibalanced in the following lemmas. All these multisets will be utilized to construct (a,d)-ASD antimagic labelings of some product graphs in Section 4.

Lemma 1. *Let $k,t\in\mathbb{N}$. The set of integers $X=[1,tk]$ is (t,k)-antibalanced.*

Proof. For $i=1,2,\ldots,t$, set

$$X_i=\{i+(j-1)t\mid j\in[1,k]\}.$$

Hence, $\sum X_i=\sum_{j=1}^{k}[i+(j-1)t]=k\left(i+\frac{t(k-1)}{2}\right)$ and $\delta=\sum X_{i+1}-\sum X_i=k$. ☐

To illustrate Lemma 1, let $t=5$ and $k=3$, and so $X=\{1,2,\ldots,15\}$. Decompose X into 5 subsets of cardinality 3: $X_1=\{1,6,11\}$, $X_2=\{2,7,12\}$, $X_3=\{3,8,13\}$, $X_4=\{4,9,14\}$, and $X_5=\{5,10,15\}$. The sumsets of the subsets are $\sum X_1=18$, $\sum X_2=21$, $\sum X_3=24$, $\sum X_4=27$, and $\sum X_5=30$. Therefore, X is $(5,3)$-antibalanced.

Lemma 2. *Let $t\in\mathbb{N}$. The set of integers $X=[1,t(t+1)]$ is (t,t^2+t+1)-ascending antibalanced.*

Proof. Set $X=\bigcup_{i=1}^{t}X_i$, where $X_i=X_{i,1}\cup X_{i,2}$, and

$$\begin{aligned}X_{i,1} &= \{(j-1)(t)-\frac{j(j-1)}{2}+i\mid j\in[1,i]\}\\ X_{i,2} &= \{t(t+2-j)+\frac{j(j-1)}{2}+1-i\mid j\in[1,i]\}.\end{aligned}$$

Then,

$$\begin{aligned}\sum X_{i,1} &= \sum_{j=1}^{i}jt-ti-\sum_{j=1}^{i}\frac{j(j-1)}{2}+i^2\\ \sum X_{i,2} &= ti(t+2)-\sum_{j=1}^{i}jt+\sum_{j=1}^{i}\frac{j(j-1)}{2}+(1-i)i.\end{aligned}$$

Therefore, $\sum X_i=\sum X_{i,1}+\sum X_{i,2}=(t^2+t+1)i$ and $\delta=\sum X_{i+1}-\sum X_i=t^2+t+1$. ☐

To illustrate Lemma 2, let $t=5$, and $X=\{1,2,\ldots,30\}$. Decompose X into 5 subsets:

$$\begin{aligned}X_1 &= \{1,30\},\\ X_2 &= \{2,6,25,29\},\\ X_3 &= \{3,7,10,21,24,28\},\\ X_4 &= \{4,8,11,13,18,20,23,27\},\ \text{and}\\ X_5 &= \{5,9,12,14,15,16,17,19,22,26\}.\end{aligned}$$

The sumsets of the subsets are $\sum X_1=31$, $\sum X_2=62$, $\sum X_3=93$, $\sum X_4=124$, and $\sum X_5=155$. Therefore, X is $(5,31)$-ascending antibalanced.

Combining the constructions in the proofs of Lemma 1 and 2 provides the following corollary.

Corollary 1. *Let* $k, t \in \mathbb{N}$. *The set of integers* $X = [1, t(t+k+1)]$ *is* $(t, t^2+t+k+1)$*-ascending antibalanced.*

Proof. Set $X = \bigcup_{i=1}^{t} X_i$, where $X_i = X_{i,1} \cup X_{i,2} \cup X_{i,3}$, for $i = 1, 2, 3, \ldots, t$. We utilize the construction in the proof of Lemma 2 for $X_{i,1}$ and $X_{i,2}$ and that in the proof of Lemma 1 for $X_{i,3}$ (by shifting to $[t^2+t+1, t(t+k+1)]$). That is,

$$\begin{aligned} X_{i,1} &= \{(j-1)(t) - \frac{j(j-1)}{2} + i \mid j \in [1,i]\} \\ X_{i,2} &= \{t(t+2-j) + \frac{j(j-1)}{2} + 1 - i \mid j \in [1,i]\}, \text{ and} \\ X_{i,3} &= \{t^2 + jt + i \mid j \in [1,k]\}. \end{aligned}$$

Then,

$$\begin{aligned} \sum X_{i,1} &= \sum_{j=1}^{i} jt - ti - \sum_{j=1}^{i} \frac{j(j-1)}{2} + i^2 \\ \sum X_{i,2} &= ti(t+2) - \sum_{j=1}^{i} jt + \sum_{j=1}^{i} \frac{j(j-1)}{2} + (1-i)i, \text{ and} \\ \sum X_{i,3} &= ki + \frac{tk(k+1)}{2} + t^2 k. \end{aligned}$$

Therefore, for $i = 1, 2, 3, \ldots, t$, $\sum X_i = (t^2+t+k+1)i + \frac{tk(k+1+2t)}{2}$. And so we obtain $\delta = \sum X_{i+1} - \sum X_i = t^2+t+k+1$. □

For example, let $t = 5$ and $k = 1$. The set of integers $X = [1, 35]$ is (5,32)-ascending antibalanced by considering the five subsets of X,

$$\begin{aligned} X_1 &= \{1, 30, 31\}, \\ X_2 &= \{2, 6, 25, 29, 32\}, \\ X_3 &= \{3, 7, 10, 21, 24, 28, 33\}, \\ X_4 &= \{4, 8, 11, 13, 18, 20, 23, 27, 34\}, \text{ and} \\ X_5 &= \{5, 9, 12, 14, 15, 16, 17, 19, 22, 26, 35\}. \end{aligned}$$

The sumsets of these subsets are $\sum X_1 = 62$, $\sum X_2 = 94$, $\sum X_3 = 126$, $\sum X_4 = 158$, and $\sum X_5 = 190$. Hence, $\delta = 32$.

The following corollary 2 is another implication of Lemma 2.

Corollary 2. *Let* $k, t, s \in \mathbb{N}$, *and* $S = \{s, \ldots, s\}$ *be a multiset containing* t *copies of* s. *Then, the multiset* $X = [1, t(t+k)] \uplus S$ *is* (t, t^2+t+k)*-ascending antibalanced.*

https://dx.doi.org/10.22108/toc.2025.143242.2219

Proof. For $k \geq 2$, set $X = \bigcup_{i=1}^{t} X_i$, where $X_i = X_{i,1} \cup X_{i,2} \cup X_{i,3} \uplus X_{i,4}$, for $i = 1, 2, 3, \ldots, t$. We utilize the set $X_{i,1}$ and $X_{i,2}$ as mentioned in the proof of Lemma 2.

$$\begin{aligned} X_{i,1} &= \{(j-1)(t) - \frac{j(j-1)}{2} + i \mid j \in [1, i]\}, \text{ and} \\ X_{i,2} &= \{t(t+2-j) + \frac{j(j-1)}{2} + 1 - i \mid j \in [1, i]\}. \end{aligned}$$

We then define $X_{i,3}$ and $X_{i,4}$ as the following:

$$\begin{aligned} X_{i,3} &= \{t^2 + jt + i \mid j \in [1, k-1]\}; \text{ and} \\ X_{i,4} &= \{s\}. \end{aligned}$$

Note that $X_{i,3} = \emptyset$ when $k = 1$. Then, we have

$$\begin{aligned} \sum X_{i,1} &= \sum_{j=1}^{i} jt - ti - \sum_{j=1}^{i} \frac{j(j-1)}{2} + i^2 \\ \sum X_{i,2} &= ti(t+2) - \sum_{j=1}^{i} jt + \sum_{j=1}^{i} \frac{j(j-1)}{2} + (1-i)i; \\ \sum X_{i,3} &= (k-1)(t^2 + i) + \frac{tk(k-1)}{2}; \text{ and} \\ \sum X_{i,4} &= s. \end{aligned}$$

Hence, $\sum X_i = (t^2 + t + k)i + \frac{(k-1)}{2} t(2t + k) + s$, for $i = 1, 2, 3, \ldots, t$ and $\delta = \sum X_{i+1} - \sum X_i = t^2 + t + k$. □

For example, let $t = 5$, $k = 1$, and $s = 31$. The multiset of integers $X = [1, 30] \uplus \{31, 31, 31, 31, 31\}$ is (5,31)-ascending antibalanced with the following five subsets of X,

$$\begin{aligned} X_1 &= \{1, 30, 31\}, \\ X_2 &= \{2, 6, 25, 29, 31\}, \\ X_3 &= \{3, 7, 10, 21, 24, 28, 31\}, \\ X_4 &= \{4, 8, 11, 13, 18, 20, 23, 27, 31\}, \text{ and} \\ X_5 &= \{5, 9, 12, 14, 15, 16, 17, 19, 22, 26, 31\}. \end{aligned}$$

The sumsets of these subsets are $\sum X_1 = 62$, $\sum X_2 = 93$, $\sum X_3 = 124$, $\sum X_4 = 155$, and $\sum X_5 = 186$. Hence, $\delta = 31$.

Lemma 3. *Let $t \in \mathbb{N}$. If $A = \{1, 1, \ldots, 1\}$ and $B = \{2, 2, \ldots, 2\}$ are the multisets containing $t - 1$ copies of 1 and 2, respectively, then the multiset $X = [1, t^2 + 2] \uplus A \uplus B$ is $(t, t^2 - t + 4)$-ascending antibalanced.*

Proof. We define $X_1 = \{1, 2, t^2+2\}$, and $X_i = X_{i,1} \cup X_{i,2} \cup X_{i,3}$, for $i = 2, 3, \ldots, t$ set, where

$$\begin{aligned} X_{i,1} &= \{1,2\}, \\ X_{i,2} &= \{(j-1)(t-1) - \frac{j(j-1)}{2} + i + 1 \mid j \in [1, i-1]\}, \text{ and} \\ X_{i,3} &= \{t(t-j+1) + \frac{j(j-1)}{2} - i + 3 \mid j \in [1, i]\}. \end{aligned}$$

Then, we obtain the following sumsets

$$\begin{aligned} \sum X_{i,1} &= 3; \\ \sum X_{i,2} &= \sum_{j=1}^{i-1}(jt - t - j + 1) - \sum_{j=1}^{i-1}\frac{j(j-1)}{2} + (i^2-1); \text{ and} \\ \sum X_{i,3} &= \sum_{j=1}^{i-1}(t^2 - jt + t) + \sum_{j=1}^{i-1}\frac{j(j-1)}{2} \\ &\quad +(t^2 - ti + t) + \frac{i(i-1)}{2} - i^2 + 3i. \end{aligned}$$

Hence, $\sum X_1 = t^2 + 5$ and $\sum X_i = (t^2 - t + 4)i + t + 1$, for $i = 2, 3, \ldots, t$. Therefore, we have $\delta = \sum X_{i+1} - \sum X_i = t^2 - t + 4$. Moreover, $|X_i| = 2i + 1$, for $i = 1, 2, \ldots, t$. This concludes that X is $(t, t^2 - t + 4)$-ascending antibalanced. □

For example, let $t = 4$. The multiset $X = \{1, 1, 1\} \uplus \{2, 2, 2\} \uplus \{1, 2, \ldots, 18\}$ can be decomposed into four submultisets $X_1 = \{1, 2, 18\}$, $X_2 = \{1, 2, 14, 17\}$, $X_3 = \{1, 2, 4, 6, 11, 13, 16\}$, and $X_4 = \{1, 2, 5, 7, 8, 9, 10, 12, 15\}$. The sumsets of these submultisets are $\sum X_1 = 21$, $\sum X_2 = 37$, $\sum X_3 = 53$, and $\sum X_4 = 69$. This shows that X is (21,16)-ascending antibalanced.

Lemma 4. *Let $t \in \mathbb{N}$ and $A = \{1, 1, \ldots, 1\}$ and $B = \{2, 2, \ldots, 2\}$ be two multisets containing $t-1$ copies of 1 and 2, respectively. The multiset $X = [1, (t+1)^2 + 1] \uplus A \uplus B$ is $(t, t^2 + t + 4)$-ascending antibalanced.*

Proof. We set $X_1 = \{1, 2, t^2 + 2\}$ and, for $i = 2, 3, \ldots, t$, $X_i = \bigcup_{j=1}^{4} X_{i,j}$, where

$$\begin{aligned} X_{i,1} &= \{1,2\}, \\ X_{i,2} &= \{(j-1)t - \frac{j(j-1)}{2} + i + 2 \mid j \in [1, i]\}, \text{ and} \\ X_{i,3} &= \{(t+1)(t-j+2) + \frac{j(j-1)}{2} - i + 2 \mid j \in [1, i+1]\}. \end{aligned}$$

Then, the sumsets of all submultisets are

$$\begin{aligned}\sum X_{i,1} &= 3,\\ \sum X_{i,2} &= \sum_{j=1}^{i}(jt-t)-\sum_{j=1}^{i}\frac{j(j-1)}{2}+(i^2+2i),\text{ and}\\ \sum X_{i,3} &= \sum_{j=1}^{i}(t^2-jt+3t-j+4-i)+\sum_{j=1}^{i}\frac{j(j-1)}{2}\\ &\quad +(t+2)(t-i)+\frac{i(i+1)}{2}+3.\end{aligned}$$

Hence, for $i = 1, 2, 3, \ldots, t$, we have $\sum X_i = (t^2 + t + 4)i + (t + 1)^2 + 5$. Therefore, we obtain the difference $\delta = \sum X_{i+1} - \sum X_i = t^2 + t + 4$. □

To illustrate this lemma, let $t = 4$. The multiset $X = \{1, 1, 1\} \uplus \{2, 2, 2, \} \uplus \{1, 2, \ldots, 26\}$ can be decomposed into four submultisets:

$$\begin{aligned}X_1 &= \{1, 2, 3, 22, 26\},\\ X_2 &= \{1, 2, 4, 7, 18, 21, 25\},\\ X_3 &= \{1, 2, 5, 8, 10, 15, 17, 20, 24\},\text{ and}\\ X_4 &= \{1, 2, 6, 9, 11, 12, 13, 14, 16, 19, 23\}.\end{aligned}$$

The sumsets of these submultisets are $\sum X_1 = 54$, $\sum X_2 = 78$, $\sum X_3 = 102$, and $\sum X_4 = 126$. This shows that X is (54,24)-ascending antibalanced.

Lemma 5. *Let $k, t \in \mathbb{N}$. If $X = [1, t(t + k) + 1]$ and $Y = [t(t + k) - (t - 2), t(t + k)]$, then $W = X \uplus Y$ is $(t, t^2 - t + k + 3)$-ascending antibalanced.*

Proof. For $i = 1$, we set $W_1 = \{t(t - 1) + (j - 1)t + 1 \mid j \in [1, k + 1]\} \cup \{t(t + k - 1) + 2\}$. Next, for $i = 2, 3, \ldots, t$, set $W_i = W_{i,1} \cup W_{i,2} \cup W_{i,3} \uplus W_{i,4}$, where

$$\begin{aligned}W_{i,1} &= \{(j-1)t-\frac{j(j+1)}{2}+i \mid j\in[1,i-1]\},\\ W_{i,2} &= \{t(t-j)+\frac{j(j+1)}{2}+1-i \mid j\in[1,i-1]\},\\ W_{i,3} &= \{t(t-1)+(j-1)t+i \mid j\in[1,k+1]\},\text{ and}\\ W_{i,4} &= \{t(t+k-1)+1+i\}.\end{aligned}$$

Then,

$$\begin{aligned}
\sum W_1 &= (t^2-t+1)k+(2t^2-3t+3)+t\frac{(k+1)(k+2)}{2},\\
\sum W_{i,1} &= (i-t)(i-1)+\frac{ti(i-1)}{2}-\sum_{j=1}^{i-1}\frac{j(j+1)}{2},\\
\sum W_{i,2} &= (t^2-i+1)(i-1)-\frac{ti(i-1)}{2}+\sum_{j=1}^{i-1}\frac{j(j+1)}{2},\\
\sum W_{i,3} &= (t^2-2t+i)(k+1)+\frac{t(k+1)(k+2)}{2}, \text{ and}\\
\sum W_{i,4} &= t(t+k-1)+1+i.
\end{aligned}$$

For $i=2,3,\dots,t$, $\sum W_i=(t^2-t+k+3)i+t^2(k+1)-t(k+2)+\frac{t(k+1)(k+2)}{2}$. Therefore, we have $\delta=\sum W_{i+1}-\sum W_i=t^2-t+k+3$ and $|W_i|=2i+k$. So, W is $(t,t^2-t+k+3)$-ascending antibalanced. □

For example, let $t=5$ and $k=1$. The multiset $X=[1,31]\uplus\{27,28,29,30\}$ is (5,24)-ascending antibalanced with the following five subsets of X,

$$\begin{aligned}
X_1 &= \{21,26,27\},\\
X_2 &= \{1,20,22,27,28\},\\
X_3 &= \{2,5,16,19,23,28,29\},\\
X_4 &= \{3,6,8,13,15,18,24,29\}, \text{ and}\\
X_5 &= \{4,7,9,10,11,12,14,17,25,30,31\}.
\end{aligned}$$

The sumsets of these subsets are $\sum X_1=74$, $\sum X_2=98$, $\sum X_3=122$, $\sum X_4=146$, and $\sum X_5=170$. Hence, $\delta=24$.

4. The (a,d)-ASD Antimagic Labelings of Product Graphs

In this section, we employ the (t,δ)-ascending antibalanced multisets constructed in Section 3 to label some (a,d)-ASD antimagic graphs. The product graphs under consideration are disjoint union, vertex amalgamation, edge amalgamation, subgraph amalgamation, and extended chain of graphs.

Let $\mathcal{H}=\{H_i\mid 1\le i\le t\}$ be a collection of disjoint graphs where $\sigma(H_i)<\sigma(H_j)$, for $1\le i<j\le t$. The disjoint union of the graphs in $\mathcal{H}$ is denoted by $\bigcup_{i=1}^{t}H_i$. We provide the (a,d)-ASD antimagic labeling for such graphs in the following.

Theorem 4.1. *Let $\mathcal{H}=\{H_i\mid i=1,2,\dots,t\}$ be a collection of graphs such that $\sigma(H_i)=2i+k$, where $k\in\{1,3,4\}$. If $G=\bigcup_{i=1}^{t}H_i$ is ASD, then G is $((t^2+t+k+1)+\frac{tk(2t+k+1)}{2},t^2+t+k+1)$-ASD antimagic.*

Proof. Since $\sigma(H_i) = 2i + k, i = 1, 2, \ldots, t$ and there is no intersection between H_i and H_j for every $i \neq j$, then G needs $\sum_{i=1}^{t}(2i+k) = t(t+k+1)$ labels of positive integers for its vertices and edges. Now, let $f : V(G) \cup E(G) \to X = \{1, 2, \ldots, t(t+k+1)\}$ be a mapping. Partition the label set $X = \bigcup_{i=1}^{t} X_i$ as mentioned in the proof of Corollary 1. To label H_i, we set $f(H_i) = X_i$ for all $i \in [1, t]$. Consequently, we can directly derive that f is a bijection with $w(H_i) = \sum X_i = (t^2 + t + k + 1)i + \frac{tk(2t+k+1)}{2}$ and $a = w(H_1) = (t^2 + t + k + 1) + \frac{tk(2t+k+1)}{2}$ as the subgraph with the smallest weight. So, we obtain $d = w(H_{i+1}) - w(H_i) = t^2 + t + k + 1$ for $i = 1, 2, \ldots, t-1$. Therefore, G is (a, d)-ASD antimagic. ☐

Note that in Theorem 4.1, $k \neq 2$, since otherwise the graph H_i will contain an isolated vertex. Furthermore, $k \leq 4$, since otherwise $\mathcal{H}$ will not satisfy the ASD requirement.

Figure 1 illustrates an example of a union of disjoint graphs that admits an (a, d)-ASD antimagic labeling. The graph is $G = \bigcup_{i=1}^{5} H_i$ where $\sigma(H_i) = 2i + 1$ and G is $(62, 32)$-ASD antimagic.

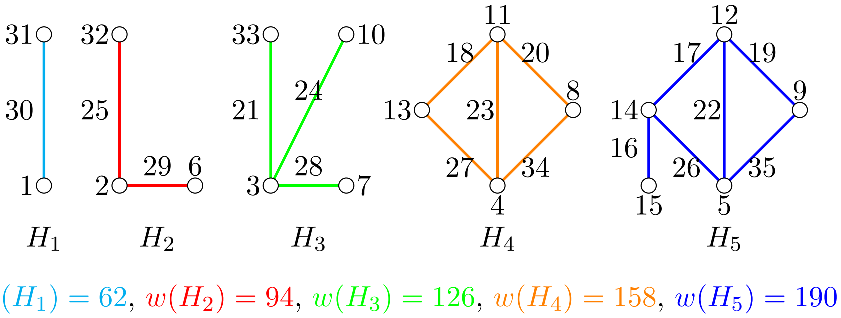

FIGURE 1. A $(62, 32)$-ASD antimagic labeling of a disjoint union of graphs

Barragán-Ramírez et al. defined the vertex, edge, and subgraph amalgamations of graphs in [2] as follows.

Definition 5. [2] *Let* $\{G_i\}$ *be a family of graphs with a common induced subgraph* J *and embedding* J_i, *where* $V(G_i) = \{u_j^i\}$, $V(J) = \{v_k\}$, *and* $V(J_i) = \{v_k^i\}$. *We define the* ***subgraph-amalgamation*** *of* $\{G_i\}$ *over* J *or* J-***amalgamation*** *of* $\{G_i\}$, *denoted by* $\amalg\{(G_i \mid J_i)\}$, *as the graph with vertex set*

$$V(\amalg\{(G_i \mid J_i)\}) = \bigcup V(G_i - J_i) \bigcup V(J)$$

and edge set

$$E(\amalg\{(G_i \mid J_i)\}) = \bigcup E(G_i - J_i) \bigcup E(J) \bigcup \{u_j^i v_k \mid v_k \in V(J) \text{ and } u_j^i v_k^i \in E(G_i)\}.$$

This means we identify the vertices in the corresponding copies of J_i and preserve the adjacencies (here, the importance of J being an induced subgraph). In the case that the commonly induced subgraph is a vertex ($J = \{v\}$) or an edge ($J = \{vw\}$), we speak about **vertex-amalgamation** or **edge-amalgamation of graph**, respectively; denoted by $v - amal\{G_i, v\}$ and $e - amal\{G_i, vw\}$, respectively.

The next theorem provides the existence of (a,d)-ASD antimagic labeling for the vertex-amalgamation of graphs. While Theorems 4.3 and 4.4 construct (a,d)-ASD antimagic labelings for vertex-amalgamation and edge-amalgamation of cycles.

Theorem 4.2. *Let $\mathcal{H} = \{H_i \mid i = 1,2,\ldots,t\}$ be a collection of graphs where $\sigma(H_i) = 2i + k$, for $i = 1,2,\ldots,t$ and $k = 1$ or $k = 3$, where each H_i contains a terminal vertex v. If $G \cong v - amal\{H_i, v\}$ is ASD, then G is $(t^2 + t + k + \frac{kt}{2}(2t - k + 1) + 1, t^2 + t + k)$-ASD antimagic.*

Proof. Since $\sigma(H_i) = 2i+k, i = 1,2,\ldots,t$, and each H_i has a mutual vertex v as terminal, then G needs $\sum_{i=1}^{t}(2i + k - 1)+1 = t(t+k)+1$ labels of positive integers. Let $t(t+k)+1$ be the label for v, which means this label will be counted t times. Thus we have $S = \{t(t+k)+1, t(t+k)+1, \ldots, t(t+k)+1\}$ and a multiset $X = \{1,2,\ldots,t(t+k)\} \uplus S$ to label the vertices and edges of G.

Let $f : V(G) \cup E(G) \to X$ be a mapping. Partition the label set $X = \bigcup_{i=1}^{t} X_i$ as mentioned in the proof of Corollary 2. To label H_i, we set $f(H_i) = X_i$ for all $i \in [1,t]$. Consequently, f is a bijection with $w(H_i) = \sum X_i = (t^2+t+k)i + \frac{kt}{2}(2t-k+1)+1$ and $a = w(H_1) = t^2+t+k+\frac{kt}{2}(2t-k+1)+1$ as the smallest weight. So, we obtain $d = w(H_{i+1}) - w(H_i) = t^2 + t + k$ for $i = 1,2,\ldots,t-1$. Therefore, G is (a,d)-ASD antimagic. □

Note that in Theorem 4.2, $k \neq 2$, since otherwise the graph H_i will contain an isolated vertex. Furthermore, $k \leq 3$, since otherwise $\mathcal{H}$ will not satisfy the ASD requirement.

For example, let $G \cong v - amal\{H_i, v\}$ with $\sigma(G) = 31$ as shown in Figure 2. G can be decomposed into 6 ascending subgraphs and labeled such that it admits a $(62,31)$-ASD antimagic labeling.

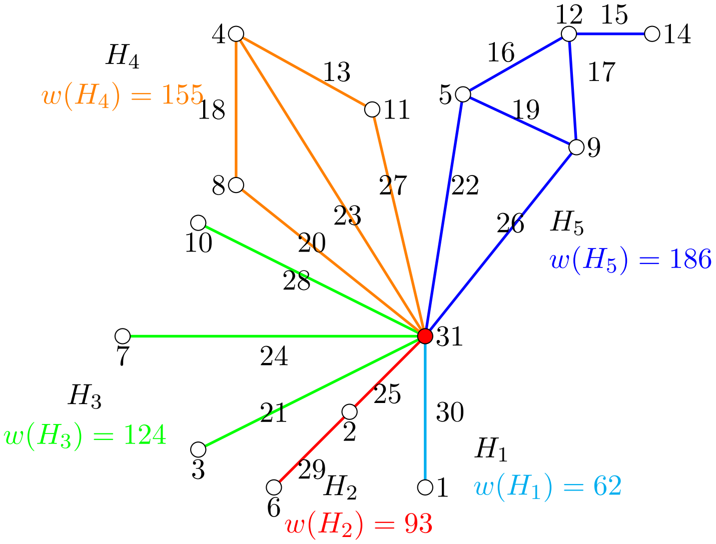

FIGURE 2. A $(62,31)$-ASD antimagic labeling of a vertex-amalgamation of graphs

Theorem 4.3. *Let $\mathcal{C} = \{C_i \mid i = 3, 4, \ldots, t+1\}$ be a collection of $t-1$ cycles with vw a common induced edge. If $G \cong e - amal\{C_i, vw\}$ is ASD, then G is (t^2+5, t^2-t+4)-ASD antimagic.*

Proof. Since each C_i has one mutual edge vw, then G needs $\sum_{i=3}^{t+1}(2i-3)+3 = t^2+2$ labels of positive integers for its vertices and edges. Define two multisets $A = \{1, 1, \ldots, 1\}$, $B = \{2, 2, \ldots, 2\}$, and construct $X = \{1, 2, \ldots, t^2+2\} \uplus A \uplus B$. Partition the label set $X = \bigcup_{i=1}^{t} X_i$ as mentioned in the proof of Lemma 3. Let $f : V(G) \cup E(G) \to X$ be a mapping, where $f(v) = 1$ and $f(w) = 2$. To label H_i, set $f(H_i) = X_i$ and $f(V(H_i)) = X_{i,1} \cup X_{i,2}$ for all $i \in [1, t]$. Consequently, f is a bijection with $w(H_i) = \sum X_i = (t^2-t+4)i+t+1$ and $a = w(H_1) = t^2+5$ as the smallest weight. So, we obtain $d = w(H_{i+1}) - w(H_i) = t^2-t+4$ for $i = 1, 2, \ldots, t-1$. Therefore, G is (t^2+5, t^2-t+4)-ASD antimagic. □

To illustrate Theorem 4.3, let $\mathcal{C} = \{C_3, C_4, C_5, C_6, \}$. An (a, d)-ASD antimagic labeling of the $e - amal\{C_i, vw\}$ graph is given in Figure 3. The common induced edge vw is indicated by the black line.

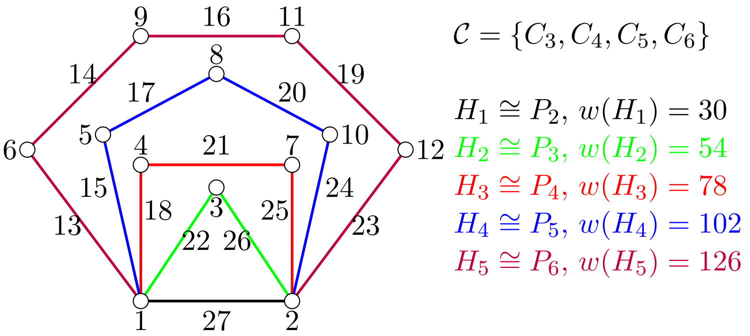

FIGURE 3. A $(30, 24)$-ASD antimagic labeling of the $e - amal\{C_i, vw\}$

Theorem 4.4. *Let $\mathcal{C} = \{C_i \mid i = 5, 6, \ldots, t+4\}$ be a collection of $t-1$ cycles, where each cycle contains a common induced subgraph $J \cong P_3$. If $G \cong \amalg\{(C_i \mid J)\}$ is an ASD graph, then G is $(2t^2+3t+10, t^2+t+4)$-ASD antimagic.*

Proof. Since each C_i in $\mathcal{C} = \{C_i \mid i = 5, 6, \ldots, t+4\}$ contains a common induced subgraph $J \cong P_3$, G needs $\sum_{i=5}^{t+4}(2i-5)+3 = (t+1)^2+1$ labels of positive integers to label its vertex set and edge set. Let v and w be the end vertices of J. Define two multisets $A = \{1, 1, \ldots, 1\}$, $B = \{2, 2, \ldots, 2\}$, and $X = \{1, 2, \ldots, (t+1)^2+1\} \uplus A \uplus B$. Partition the label set $X = \bigcup_{i=1}^{t} X_i$ as mentioned in the proof of Lemma 4. Let $f : V(G) \cup E(G) \to X$ be a mapping where $f(v) = 1$ and $f(w) = 2$. To label H_i, we set $f(V(H_i)) = X_{i,1} \cup X_{i,2}$, for all $i \in [1, t]$. Therefore, we obtain that f is bijection, with the weight of each subgraph $w(H_i) = \sum X_i = (t^2+t+4)i+(t+1)^2+5$ and $a = w(H_1) = 2t^2+3t+10$ is the smallest weight. Moreover, we obtain the difference $d = w(H_{i+1}) - w(H_i) = t^2+t+4$ for $i = 1, 2, \ldots, t-1$. This concludes that G is $(2t^2+3t+10, t^2+t+4)$-ASD antimagic. □

To illustrate Theorem 4.4, let $\mathcal{C} = \{C_5, C_6, C_7\}$, where each C_i contains $J \cong P_3$. An (a,d)-ASD antimagic labeling of $\amalg\{(C_i \mid P_3)\}$ is given in Figure 4. The common induced subgraph $J \cong P_3$ is indicated by the green path.

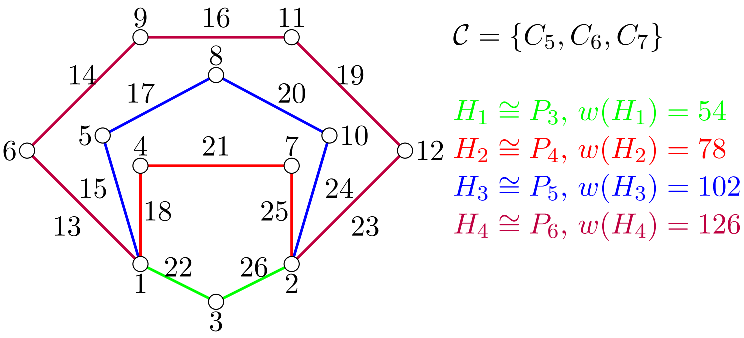

FIGURE 4. A $(54, 24)$-ASD antimagic labeling of the subgraph amalgamation of cycles

The last graph product under consideration is the extended chain graph. This product was first introduced as the chain graph in 2002 [3] and the extended version was defined in 2017 [4], both by Barrientos.

Definition 6. [4] *Suppose that $\mathcal{G} = \{G_i \mid 1 \leq i \leq t\}$ is an ordered collection of connected graphs and, for every $1 \leq i \leq t$, let $u_i, v_i \in V(G_i)$. An* ***extended chain graph*** *of $\mathcal{G}$ is a graph resulting by identifying the vertex v_i with u_{i+1}, for each $1 \leq i \leq t-1$. The collection of extended chain graphs of $\mathcal{G}$ is denoted by $\mathcal{E}xChain\{G_i\}_{i=1}^{t}$.*

Note that an extended chain graph of $\mathcal{G}$ can vary depending on the choice of u_i and v_i for each G_i. The existence of (a,d)-ASD antimagic labelings for the extended chain graph is discussed in the next theorem.

Theorem 4.5. *Let $\mathcal{H} = \{H_i \mid i = 1, 2, \ldots, t\}$ be an ordered collection of connected graphs such that $\sigma(H_i) = 2i + k$, where $k = 1$ or $k = 3$. If $G \in \mathcal{E}xChain\{H_i\}$ is ASD, then G is $((t^2 - t + 1)k + (2t^2 - 3t + 3) + t\frac{(k+1)(k+2)}{2}, t^2 - t + k + 3)$-ASD antimagic.*

Proof. Consider that $\sigma(H_i) = 2i + k, i = 1, 2, \ldots, t$, where $k = 1$ or 3. For every $x, y \in [1, t]$, where $|x - y| \geq 2$, H_x and H_y do not have a common vertex, while for every $i \in [1, t-1]$, H_i and H_{i+1} have exactly one common connecting vertex. In this case, there will be $t - 1$ labels that are used twice, and H needs $t(t + k) + 1$ labels for its vertices and edges.

Construct a multiset of labels $W = [1, t(t + k) + 1] \uplus [t(t + k) - (t - 2), t(t + k)]$ and let $f : V(G) \cup E(G) \to W$ be mapping. Using Lemma 5, label the $t - 1$ connecting vertices using the elements of $W_{i,4}$. To label H_i, we set $f(H_i) = W_i$. As a result, f is a bijection, with the weight of each subgraph $w(H_i) = (t^2 - t + k + 3)i + t^2(k + 1) - t(k + 2) + \frac{t(k+1)(k+2)}{2}$ and the smallest weight is

$a = w(H_1) = (t^2-t+1)k+(2t^2-3t+3)+t\frac{(k+1)(k+2)}{2}$. Therefore, $d = w(H_{i+1})-w(H_i) = t^2-t+k+3$ and G is (a,d)-ASD antimagic. □

Note that in Theorem 4.5, $k \neq 2$, since otherwise the graph H_i will contain an isolated vertex. Furthermore, $k \leq 3$, since otherwise $\mathcal{H}$ will not satisfy the ASD requirement.

Figure 5 illustrates an extended chain graph with six ascending subgraphs that is (a,d)-ASD antimagic.

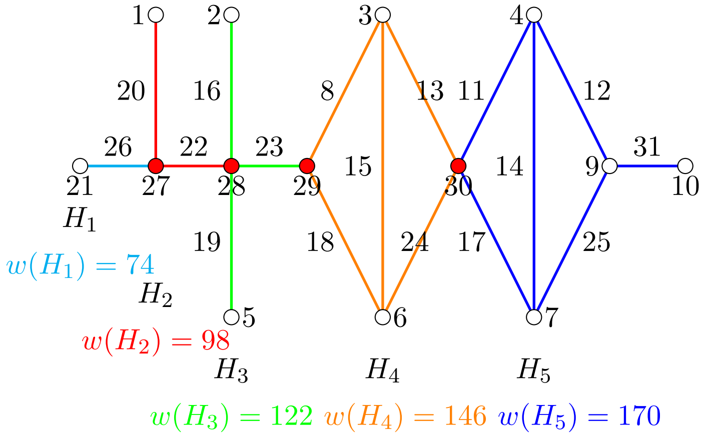

FIGURE 5. A $(74,24)$-ASD antimagic labeling of an extended chain graph

Note that the product graphs of Theorems 4.1, 4.2, 4.3, 4.4, and 4.5 are required to be ASD. The following two theorems provide examples where such requirements are fulfilled.

Theorem 4.6. *Let G be an ASD graph with $\mathcal{H} = \{H_i \mid 1 \leq i \leq t\}$ as its ascending subgraphs collection. If*

(1) $G' = \bigcup_{i=1}^{t} H_i$,
(2) $G' = v - amal\{H_i, v\}$,
(3) $G' \in \mathcal{E}xChain\{H_i\}_{i=1}^{t}$,

then G' is ASD.

Proof.

(i) Since $\mathcal{H}$ is collection of ascending subgraphs, the disjoint union $\bigcup_{i=1}^{t} H_i$ maintains the structure of every H_i, that is for $1 \leq i \leq t-1$, H_i is isomorphic to a proper subgraph of H_{i+1}. Therefore, $G' = \bigcup_{i=1}^{t} H_i$ is an ASD graph.

(ii) Since v belongs to every H_i, $G' = \mathcal{A}mal\{H_i, v\}$ can be decomposed into t ascending subgraphs $H_1, H_2, \ldots, H_t$ where H_i is isomorphic to a proper subgraph of H_{i+1} for $1 \leq i \leq t-1$. Therefore, $G' = v - amal\{H_i, v\}$ is an ASD graph.

(iii) Let $c_1, c_2, \dots, c_{t-1}$ are connecting vertices and each c_i replaces the two embedding vertices v_i and u_{i+1}. To decompose $G' = \mathcal{E}xChain\{H_i\}_{i=1}^{t}$ into t ascending subgraphs, set $H'_i \cong H_i$ for every $1 \leq i \leq t$ and set c_i to be a vertex in both $V(H_i)$ and $V(H_{i+1})$. Thus we obtain $\mathcal{H}' = \{H'_i \mid 1 \leq i \leq t\}$ is a collection of ascending subgraphs of G' and therefore G' is ASD.

□

Theorem 4.7.

(i) *Let $\mathcal{C} = \{C_i \mid 3 \leq i \leq t+1\}$ be a collection of $t-1$ cycles, where each cycle has a common edge e. Then $G' = e - amal\{C_i, e\}$ is ASD.*

(ii) *Let $\mathcal{C} = \{C_i \mid 5 \leq i \leq t+4]\}$ be a collection of $t-1$ cycles, where each cycle has a common induced subgraph $J \cong P_3$. Then $G' = \amalg\{(C_i \mid P_3)\}$ is ASD.*

Proof.

(i) Since every C_i has a common edge e, $G' = e - amal\{C_i, e\}$ can be decomposed into t ascending subgraphs $H_i \cong P_{i+1}$, for $1 \leq i \leq t$. This implies that H_i is isomorphic to a proper subgraph of H_{i+1}. Therefore, $G' = e - amal\{C_i, e\}$ is an ASD graph.

(ii) Since every C_i has a common induced subgraph $J \cong P_3$, $G' = \amalg\{(C_i \mid P_3)\}$ can be decomposed into t ascending subgraphs $H_i \cong P_{i+2}$, for $1 \leq i \leq t$. This implies that H_i is isomorphic to a proper subgraph of H_{i+1}. Therefore, $G' = \amalg\{(C_i \mid P_3)\}$ is an ASD graph.

□

5. Conclusion and Open Problems

Previously in Section 2, we provided an upper bound for the values of the difference d of an (a, d)-ASD antimagic graph (Theorem 2.1). We compare the upper bounds and the actual values of d for all (a, d)-ASD antimagic product graphs studied in Section 4 and present them in Table 1. It is obvious that the values of d are much less than the upper bounds, which suggests the problem of finding a better upper bound.

TABLE 1. Upper bounds for d of (a, d)-ASD antimagic product graphs

product graph	d	upper bound for d
$G = \bigcup_{i=1}^{t} H_i$	$t^2 + t + k + 1$, for $k \in \{1, 3, 4\}$	$\frac{(2t+k)[4t^2+2t+2tk-k+1]}{2(t-1)} - \frac{6+5k+k^2}{2(t-1)}$
$G = v - amal\{H_i, v\}$	$t^2 + t + k$, for $k \in \{1, 3\}$	$\frac{(2t+k)[2t^2+2kt-3k+3]}{2(t-1)} - \frac{(6+5k+k^2)}{2(t-1)}$
$G = e - amal\{C_{i+2}\}_{i=1}^{t}$	$t^2 - t + 4$	$2t^2 + t + 4$
$G = \amalg\{(C_i \mid P_3)\}$	$t^2 + t + 4$	$2t^2 + 7t + 12$
$G = \mathcal{E}xChain\{H_i\}_{i=1}^{t}$	$t^2 - t + k + 3$, for $k \in \{1, 3\}$	$\frac{(2t+k)(2t^2+2tk-2t-k+3)}{2(t-1)} - \frac{(6+5k+k^2)}{2(t-1)}$

https://dx.doi.org/10.22108/toc.2025.143242.2219

In this paper, we introduce the notion of (a, d)-ASD antimagic labeling, which arises from the ascending subgraph decomposition. Our preliminary results provide some (a, d)-ASD antimagic product graphs. For further research, we ask the following questions connected to (a, d)-ASD antimagic labelings of graphs.

Problem 1. *Find a better upper bound for the difference* d *of an* (a, d)*-ASD antimagic graph.*

Problem 2. *Find a "good" lower bound for the difference* d *of an* (a, d)*-ASD antimagic graph.*

Problem 3. *Let* G *be an ASD graph, with* $\mathcal{H} = \{H_i \mid i = 1, 2, \ldots, t\}$ *the collection of ascending subgraphs. Find other graph products of* $\mathcal{H}$ *that preserve the ASD property and decide whether the product graphs are* (a, d)*-ASD antimagic.*

Acknowledgments

This research has been partially supported by Penelitian Kerja Sama Dalam Negeri 2023 (Number 0536/E5/PG.02.00/2023) funded by the Directorate General of Higher Education, Research, and Technology, the Ministry of Education, Culture, Research, and Technology. The first author was supported by LPDP Scholarship funded by the Ministry of Finance. The authors are thankful to the anonymous referee for some comments that helped to improve the presentation of the manuscript.

REFERENCES

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdös, and O. R. Oellermann, The ascending subgraph decomposition problem, *Congr. Numer.*, **58** (1987) 7–14.

[2] G. A. Barragán-Ramírez, R. Simanjuntak, S. W. Saputro and S. Uttunggadewa, The local metric dimension of subgraph-amalgamation of graphs, (2015), arXiv:1512.07420v1. https://doi.org/10.48550/arXiv.1512.07420

[3] C. Barrientos, Graceful labelings of chain and corona graphs, *Bull. Inst. Combin. Appl.*, **34** (2002) 17–26.

[4] C. Barrientos and S. Minion, Snakes: from graceful to harmonious, *Bull. Inst. Combin. Appl.*, **79** (2017) 95–107.

[5] P. Erdös, A. Goodman and L. Posa, The representation of a graph by set intersection, *Canadian J. Math.*, **18** (1966) 106–112.

[6] N. Inayah, A. Lladó and J. Moragas, Magic and antimagic H-decomposition, *Discrete Math.*, **312** no. 7 (2012) 1367–1371.

[7] N. Inayah, A. N. M. Salman and R. Simanjuntak, On (a, d)-H-antimagic coverings of graphs, *J. Combin. Math. Combin. Comput.*, **71** (2009) 273–281.

[8] N. Inayah, R. Simanjuntak, A. N. M. Salman, and K. I. A. Syuhada, Super (a, d)-H-antimagic total labelings for shackles of a connected graph H, *Australas. J. Combin.*, **57** (2013) 127–138.

[9] Z. Liang and H. L. Fu, On ascending subgraph decomposition of graphs, *J. Discrete Math. Sci. Cryptogr.*, **20** no. 5 (2017) 1135–1149.

[10] J. Petersen, Die theorie der regulären graphs, *Acta Math.*, **15** no. 1 (1891) 193–220.

Sigit Pancahayani

Doctoral Program in Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia

Email: `30122004@mahasiswa.itb.ac.id`

Rinovia Simanjuntak

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia

and

Centre for Research Collaboration in Graph Theory and Combinatorics, Indonesia

Email: `rino@itb.ac.id`

Saladin Uttunggadewa

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia

Email: `s_uttunggadewa@math.itb.ac.id`

https://dx.doi.org/10.22108/toc.2025.143242.2219