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ON WIENER INDEX OF GRAPH COMPLEMENTS

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Abstract. Let G be an (n, m)-graph. We say that G has property $(*)$ if for every pair of its adjacent vertices x and y, there exists a vertex z, such that z is not adjacent to either x or y. If the graph G has property $(*)$, then its complement $\overline{G}$ is connected, has diameter 2, and its Wiener index is equal to $\binom{n}{2} + m$, i.e., the Wiener index is insensitive of any other structural details of the graph G. We characterize numerous classes of graphs possessing property $(*)$, among which are trees, regular, and unicyclic graphs.

1. Introduction

Let G be a simple graph with vertex set $V(G)$. If u and v are vertices of G, then the distance between them is the length of (= number of edges in) a shortest path connecting u and v [1,3]. If the vertices u and v belong to different components of G, then their distance is not defined.

The greatest distance between two vertices in a connected graph is said to be the diameter of this graph.

For a connected graph G, the sum of the distances between all pairs of vertices is the Wiener index, and will be denoted by $W = W(G)$. Mathematical studies of the Wiener index started in the 1970s [7]. Since then, numerous properties of W were established, and scores of papers on this graph invariant published; for details see the reviews [6, 4, 5, 14] and the recent papers [2, 8, 11, 12]. In addition, the Wiener index found remarkable applications in chemistry [10, 13].

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The complement of the graph G is the graph with vertex set $V(G)$, in which two vertices are adjacent if and only if they are not adjacent in G. As usual, the complement of the graph G will be denoted by $\overline{G}$.

Interestingly, in spite of the half–a–century long research on the Wiener index, results pertaining to graph complements seem to be missing. The aim of this paper is to contribute towards filling this gap. We demonstrate the validity of a somewhat surprising result: For large classes of graphs, $W(\overline{G})$ depends only on the number of vertices and edges of the graph G.

The present paper is organized as follows. In Section 2, we state a simple lemma, enabling the calculation of the Wiener index of complements of graphs that satisfy a certain condition. In Section 3, we point out a number of corollaries of this lemma.

2. A simple lemma

A graph with n vertices and m edges will be referred to as an (n,m)-graph.

Definition 2.1. *We say that the graph G has property $(*)$ if for any two of its adjacent vertices x and y, there exists a third vertex z which is not adjacent to either x or y.*

Lemma 2.2. *Let G be an (n,m)-graph. If G has property $(*)$, then*
(a) $\overline{G}$ is connected,
(b) the diameter of $\overline{G}$ is two, and
(c) the Wiener index of $\overline{G}$ satisfies the identity

$$W(\overline{G}) = \binom{n}{2} + m \ . \tag{2.1}$$

Proof. Let $u, v \in V(G)$. If u and v are not adjacent in G, then they are adjacent in $\overline{G}$ and thus their distance in $\overline{G}$ is 1. If u and v are adjacent in G, then by property $(*)$ there is a vertex $w \in V(G)$, adjacent to neither u nor v. Then in $\overline{G}$, the vertices u and v are not adjacent, but u and w, as well as v and w are adjacent. Consequently, the distance between u and v in $\overline{G}$ is 2. Since u and v are any two vertices of G, it follows that $\overline{G}$ is connected and, furthermore, the greatest distance in it is 2. This proves claims (a) and (b).

In $\overline{G}$ there are $\binom{n}{2} - m$ pairs of adjacent vertices (at distance 1) and the remaining m pairs of vertices are at distance 2. Thus,

$$W(\overline{G}) = 1 \cdot \left[\binom{n}{2} - m \right] + 2 \cdot m$$

and Eq. (2.1) follows. By the way, the expression for the Wiener index of graphs with diameter 2 is previously known [7, 9]. □

3. Corollaries of Lemma 2.2

Theorem 3.1. *Let G be an (n,m)-graph. If G is disconnected, then Eq. (2.1) holds.*

Proof. Property $(*)$ evidently holds. □

The star S_n is the tree of order n, possessing $n-1$ pendent vertices. For $a \geq 1$, $b \geq 1$, $a+b=n-2$, the double-star $DS_{a,b}$ is the tree of order n, possessing $n-2$ pendent vertices, a vertex of degree $a+1$ and a vertex of degree $b+1$. Note that the diameter of S_n is two, whereas the diameter of $DS_{a,b}$ is three.

By P_n we denote the n-vertex path, i.e., the tree of order n with exactly two pendent vertices. Its diameter is $n-1$.

Theorem 3.2. *Let $n \geq 5$, and let T be a tree of order n which is neither the star nor a double-star. Then*

$$W(\overline{T}) = \frac{1}{2}\,(n-1)(n+2)\ . \tag{3.1}$$

Proof. By direct checking we can verify that the path P_5 has property $(*)$, i.e., for any two adjacent vertices of P_5 , there is a third non-adjacent vertex.

A tree T which is neither a star nor a double-star has diameter greater than 3. Therefore, any two adjacent vertices of T belong to a P_5 spanning subgraph of T. Therefore, T has property $(*)$. Eq. (3.1) follows then from Eq. (2.1), because $m=n-1$. □

The complement of a star is disconnected and thus has no Wiener index. By an easy calculation we obtain

$$W(\overline{DS}_{a,b}) = \binom{a+b}{2} + 3(a+b+1) = \binom{n-2}{2} + 3(n-1)\ .$$

If $2 \leq n \leq 4$, then the only n-vertex tree whose complement is connected is P_4 . Note that $\overline{P_4} \cong P_4$ and $W(\overline{P_4}) = 10$.

Theorem 3.3. *Let R be a regular graph of order n and of degree r. If $r \leq (n-1)/2$, then*

$$W(\overline{R}) = \frac{1}{2}\,n(n+r-1)\ . \tag{3.2}$$

Proof. Two adjacent vertices x,y of R are adjacent to at most $2r-2$ other vertices. Therefore, if $n-[(2r-2)+2] \geq 1 \iff r \leq (n-1)/2$, then there will be at least one vertex z not adjacent to x and y, implying property $(*)$. Eq. (3.2) follows from Eq. (2.1) bearing in mind that $m=\frac{1}{2}\,nr$. □

Using the notation specified in Theorem 3.3, we also have the following. If $r \geq n-2$, then $\overline{R}$ is disconnected and its Wiener index does not exist. If $(n-1)/2 < r < n-2$, then $\overline{R}$ may be either connected or disconnected; if $(n-1)/2 < r < n-3$ and $\overline{R}$ is connected, then $W(\overline{R})$ depends on the actual structure of the graph R and is not determined solely by the parameters n and r.

The next result is a special case of Theorem 3.3, for $r=2$.

Theorem 3.4. *Let C_n be the cycle of order n. If $n \geq 5$, then*

$$W(\overline{C}_n) = \binom{n+1}{2} .$$

Recall that the complements of C_3 and C_4 are disconnected, thus having no Wiener index.

A direct extension of Theorem 3.4 is:

Theorem 3.5. *Let G be an (n,m)-graph with girth g. If $g \geq 5$, then Eq. (2.1) holds.*

Let U be a connected unicyclic graph of order n (thus possessing n edges) and girth g. Theorems 3.4 and 3.5 imply that if $g \geq 5$, then

$$W(\overline{U}) = \binom{n+1}{2} . \tag{3.3}$$

We now turn our attention to the cases $g = 3$ and $g = 4$.

Let a and b be non-negative integers. Let the graph $C_3(a,b)$ be obtained by attaching a pendent vertices to a vertex of C_3 and by attaching b pendent vertices to another vertex of C_3 . Let the graph $C_4(a,b)$ be obtained by attaching a pendent vertices to a vertex u of C_4 and by attaching b pendent vertices to a first neighbor of u.

Theorem 3.6. *Eq. (3.3) holds for all connected unicyclic graphs U of order n, except for the graphs $C_3(a,b)$ and $C_4(a,b)$, $a, b \geq 0$.*

Proof. It is easy to verify that $C_3(a,b)$ and $C_4(a,b)$ are the only connected unicyclic graphs failing to have property $(*)$. □

Concluding this list of examples of graphs for which Lemma 2.2 is applicable, we state without proof:

Theorem 3.7. *Let G be an (n,m)-graph. Let G^+ be the graph obtained by attaching a pendent vertex to each vertex of G. Then*

$$W(\overline{G^+}) = \binom{2n}{2} + n + m$$

except if G is connected and $n \leq 2$.

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