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CONNECTED GRAPHS COSPECTRAL WITH A FRIENDSHIP GRAPH

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Abstract. Let n be any positive integer, the friendship graph F_n consists of n edge-disjoint triangles that all of them meeting in one vertex. A graph G is called cospectral with a graph H if their adjacency matrices have the same eigenvalues. Recently in http://arxiv.org/pdf/1310.6529v1.pdf it is proved that if G is any graph cospectral with F_n ($n \neq 16$), then $G \cong F_n$. Here we give a proof of a special case of the latter: Any connected graph cospectral with F_n is isomorphic to F_n. Our proof is independent of ones given in http://arxiv.org/pdf/1310.6529v1.pdf and the proofs are based on our recent results given in [*Trans. Comb.*, **2** no. 4 (2013) 37-52.] using an upper bound for the largest eigenvalue of a connected graph given in [*J. Combinatorial Theory Ser. B* **81** (2001) 177-183.].

1. **Introduction**

The friendship graph F_n is the graph consisting of n edge-disjoint cycles of length 3, all meeting in a common vertex (see Figure 1). The adjacency spectrum $Spec(G)$ of a graph G is the multiset of eigenvalues of its adjacency matrix. If distinct eigenvalues of the graph G are $\lambda_1, \dots, \lambda_k$ with multiplicities $m_1, \dots, m_k$ respectively, then we denote $Spec(G)$ by $\{\lambda_1^{m_1}, \dots, \lambda_k^{m_k}\}$. A graph G is called determined by its adjacency spectrum (for short DS), if $Spec(G) = Spec(H)$ for some graph H, then $G \cong H$. In [5, 6], it is conjectured that the friendship graph is DS. In [3], it is claimed that the conjecture is valid, but it is noted in [1] that the proof has a flaw. In [1] some results on graphs cospectral with F_n has been obtained. Finally in [2] a more general result about graphs with all but two eigenvalues equal to ± 1 is obtained and as a corollary it is shown that F_n is DS if $n \neq 16$ and F_{16} is cospectral with the disjoin union of 10 complete graph K_2 with the graph given in Figure 2. The spectrum of the graph of Figure 2 is $\left\{\left(1 \pm \sqrt{129}\right)/2, 1^5, -1^6\right\}$. Our main result is the following:

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Theorem 1.1. *Any connected graph cospectral with F_n is isomorphic to F_n.*

The proof is based on our previous recent results in [1] by using an upper bound given in [4] for the spectral radius of a connected graph in terms of the minimum degree of the graph and its number of vertices and edges. The proof is independent of that of given in [2].

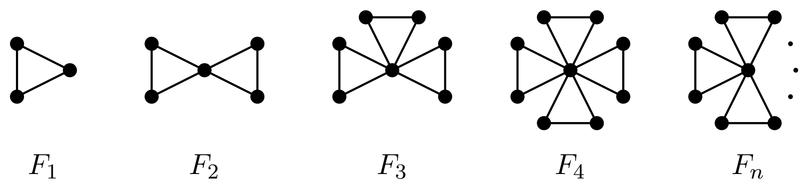

FIGURE 1. Friendship graphs F_1, F_2, F_3, F_4 and F_n

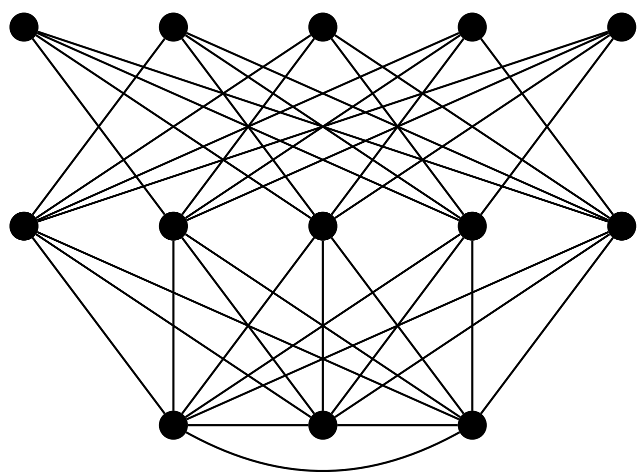

FIGURE 2. The disjoint union of the above graph with $10K_2$ is cospectral with F_{16}

2. Proof of Theorem 1.1

In the following, we give results that we will need in the proof of Theorem 1.1.

Proposition 2.1. [1, Proposition 2.3] *Let F_n denote the friendship graph with $2n+1$ vertices. Then*

$$Spec(F_n) = \left\{ \left(\frac{1}{2} - \frac{1}{2}\sqrt{1+8n}\right)^1, (-1)^n, (1)^{n-1}, \left(\frac{1}{2} + \frac{1}{2}\sqrt{1+8n}\right)^1 \right\}.$$

The maximum eigenvalue of a graph G is called spectral radius and it is denoted by $\varrho(G)$.

Lemma 2.2. [1, Lemma 3.2] *Let G be a connected graph that is cospectral with F_n and $\delta(G)$ be the minimum degree of G. Then, $\delta(G) = 2$ and G has at least $1 + \varrho(F_n)$ vertices with this minimum degree.*

Theorem 2.3. [1, Theorem 3.11] *Let G be a graph cospectral with F_n and G has two adjacent vertices of degree 2. Then G is isomorphic to F_n.*

A graph is called bidegreed if the set of degrees of vertices consists of exactly two distinct elements.

Theorem 2.4. [4, Theorem 2.3] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Now we are ready to prove the main result.

Proof of Theorem 1.1. Let G be a connected graph and cospectral with the friendship graph F_n. By Proposition 2.1, $\varrho(G) = \frac{1}{2} + \frac{1}{2}\sqrt{1+8n}$ and the number of vertices and edges of G are the same as F_n. Also by Lemma 2.2, $\delta = \delta(G) = 2$. Therefore, the equality in Theorem 2.4 holds and so G is either a regular graph or a bidegreed graph in which each vertex is of degree either 2 or $2n$. If $n = 1$, then it is easy to see that $G \cong F_1 = K_3$. If $n > 1$ then F_n is not regular and so G is not a regular graph, since regularity can be determined by the adjacency spectrum of a graph. Hence every vertex of G has degree 2 or $2n$. Since G has $2n+1$ vertices and $3n$ edges, it follows that G has only one vertex of degree $2n$ and all other vertices has degree 2. Thus G has at least two adjacent vertices of degree 2. Now Theorem 2.3 completes the proof. □

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