www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 2 (2014), pp. 21-29.

www.ui.ac.ir

MULTIPLICATIVE ZAGREB ECCENTRICITY INDICES OF SOME COMPOSITE GRAPHS

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Communicated by Ali Reza Ashrafi

Abstract. Let G be a connected graph. The multiplicative Zagreb eccentricity indices of G are defined respectively as $\mathbf{\Pi}_1^*(G) = \prod_{v\in V(G)} \varepsilon_G^2(v)$ and $\mathbf{\Pi}_2^*(G) = \prod_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v)$, where $\varepsilon_G(v)$ is the eccentricity of vertex v in graph G and $\varepsilon_G^2(v) = (\varepsilon_G(v))^2$. In this paper, we present some bounds of the multiplicative Zagreb eccentricity indices of Cartesian product graphs by means of some invariants of the factors and supply some exact expressions of $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of some composite graphs, such as the join, disjunction, symmetric difference and composition of graphs, respectively.

1. Introduction

Throughout this paper we consider only undirected simple connected graphs. For terminology and notations are not defined here we refer the reader to West [10].

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, the number of vertices and edges of G will be denoted by $|V(G)|$ (or $|G|$) and $|E(G)|$, respectively. We denote the degree and the neighborhood of a vertex v of G by $\deg_G(v)$ and $N_G(v)$, then $\deg_G(v) = |N_G(v)|$. As usual the distance between vertices u and v of graph G, denoted by $d_G(u, v)$, is defined as the number of edges in a shortest path connecting the vertices u and v. For a vertex v of $V(G)$, its eccentricity $\varepsilon_G(v)$ is the largest distance between v and any other vertex u of G, i.e., $\varepsilon_G(v) = \max_{u\in V(G)} d_G(u, v)$. A vertex $v \in V(G)$ is well-connected if $\varepsilon_G(v) = 1$, i.e., if it is adjacent to all other vertices of G. We suppose that W denotes the set of all well-connected vertices of G and W_i is the set of all well-connected vertices of G_i $(i = 1, 2, \ldots, n)$, respectively.

MSC(2010): Primary: 05C12; Secondary: 05C76.

Keywords: Multiplicative Zagreb eccentricity indices; composite operations; Cartesian product.

Received: 18 October. 2013, Accepted: 5 April 2014.

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A topological index is a real number related to a (molecular) graph, it does not rely on the labeling or the pictorial representation of a graph. There are several topological indices be defined in the chemical literatures, especially those vertex-degree-based and vertices-distance-based topological indices, which have been found many applications as means for modeling chemical, pharmaceutical and other properties of molecules.

One of the oldest and extensively studied vertex-degree-based topological indices are the first and second Zagreb indices M_1 and M_2, introduced by Gutman and Trinajstić [5], which are applied to study molecular, chirality in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) analysis, etc. They are defined as $M_1(G) = \sum_{v\in V(G)}(\deg_G(v))^2$ and $M_2(G) = \sum_{uv\in E(G)} \deg_G(u)\deg_G(v)$. The multiplicative variant of Zagreb indices was introduced by Todeschini et. al. [8], they are defined as $\mathbf{\Pi}_1(G) = \prod_{v\in V(G)}(\deg_G(v))^2$ and $\mathbf{\Pi}_2(G) = \prod_{uv\in E(G)} \deg_G(u)\deg_G(v)$. Gutman in [6] call these indices as multiplicative Zagreb indices.

Recently, the topological indices based on vertex eccentricities attracted some attention in Chemistry. In analogy with the ordinary Zagreb indices, the first and second Zagreb eccentricity indices M_1^* and M_2^* of a connected graph G have been introduced by Ghorbani et al. [4] and Vukičević et al. [9] as the revised version of the Zagreb indices. They are defined as $M_1^*(G) = \sum_{v\in V(G)}(\varepsilon_G(v))^2$ and $M_2^*(G) = \sum_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v)$. Ghorbani and Hosseinzadeh computed the Zagreb eccentricity indices of some composite graphs. Vukičević and Graovac showed that $M_1^*(G)/|G| \geq M_2^*(G)/|E(G)|$ holds for all acyclic and unicyclic graphs and that neither this nor the opposite inequality holds for all bicyclic graphs. For further results of the Zagreb eccentricity indices, we encourage the reader refer to [1, 3, 11].

In analogy with the first and second multiplicative Zagreb indices, the multiplicative Zagreb eccentricity indices of the connected graph G were introduced by De [2] and are defined respectively as

$$\begin{aligned}
\mathbf{\Pi}_1^*(G) &= \prod_{v\in V(G)} \varepsilon_G^2(v) = \prod_{v\in V(G)} (\varepsilon_G(v))^2, \\
\mathbf{\Pi}_2^*(G) &= \prod_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v).
\end{aligned}$$

In this paper, we present some bounds of the multiplicative Zagreb eccentricity indices of Cartesian product graphs by means of some invariants of the factors, and supply some exact expressions of $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of some composite graphs, such as the join, disjunction, symmetric difference and composition of graphs, respectively.

2. Some bounds of $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of Cartesian product graphs

The Cartesian product $G_1\Box G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1\Box G_2) = V(G_1)\times V(G_2)$ and $(u_1, u_2)(v_1, v_2)$ is an edge of $G_1\Box G_2$ if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. If $G_1, G_2, \ldots, G_n$ are $n(\geq 2)$ graphs, then we denote $G_1\Box G_2\Box\cdots\Box G_n$ by $\Box_{i=1}^n G_i$.

Lemma 2.1. [4] *Let $G_1, G_2, \ldots, G_n$ be n graphs. Then*

$$\begin{aligned} \deg_{\square_{i=1}^n G_i}((u_1, u_2, \ldots, u_n)) &= \sum_{i=1}^n \deg_{G_i}(u_i), \\ \varepsilon_{\square_{i=1}^n G_i}((u_1, u_2, \ldots, u_n)) &= \sum_{i=1}^n \varepsilon_{G_i}(u_i). \end{aligned}$$

Theorem 2.2. *Let $G_1, G_2, \ldots, G_n$ be n nontrivial graphs. Then*

$$\mathbf{\Pi}_1^*(\square_{i=1}^n G_i) \geq n^{2\prod_{i=1}^n |G_i|} \times \prod_{i=1}^n \big(\mathbf{\Pi}_1^*(G_i)\big)^{\frac{1}{n}\prod_{j=1, j\neq i}^n |G_j|}$$

with equality if and only if $\varepsilon_{G_1}(u_1) = \varepsilon_{G_2}(u_2) = \cdots = \varepsilon_{G_n}(u_n)$ for any vertex $(u_1, u_2, \ldots, u_n)$ in $\square_{i=1}^n G_i$.

Proof. By *Arithmetic-Geometric Mean Inequality*, we have

$$\sum_{i=1}^n \varepsilon_{G_i}(u_i) \geq n\sqrt[n]{\prod_{i=1}^n \varepsilon_{G_i}(u_i)} = n\big(\prod_{i=1}^n \varepsilon_{G_i}(u_i)\big)^{\frac{1}{n}}. \tag{2.1}$$

So, from Lemma 2.1 and the inequality (2.1), we can obtain that

$$\begin{aligned} \mathbf{\Pi}_1^*(\square_{i=1}^n G_i) &= \prod_{(u_1,u_2,\ldots,u_n)\in V(\square_{i=1}^n G_i)} \varepsilon^2_{\square_{i=1}^n G_i}((u_1, u_2, \ldots, u_n)) \\ &= \prod_{(u_1,u_2,\ldots,u_n)\in V(\square_{i=1}^n G_i)} \big(\sum_{i=1}^n \varepsilon_{G_i}(u_i)\big)^2 \\ &\geq \prod_{(u_1,u_2,\ldots,u_n)\in V(\square_{i=1}^n G_i)} n^2\sqrt[n]{\prod_{i=1}^n \varepsilon^2_{G_i}(u_i)} \\ &= n^{2\prod_{i=1}^n |G_i|} \times \prod_{i=1}^n \big(\mathbf{\Pi}_1^*(G_i)\big)^{\frac{1}{n}\prod_{j=1, j\neq i}^n |G_j|}. \end{aligned}$$

The proof is completed. □

Theorem 2.3. *Let $G_i (i = 1, 2, \ldots, n)$ does not contain any well-connected vertex. Then*

$$\mathbf{\Pi}_1^*(\square_{i=1}^n G_i) \leq \prod_{i=1}^n \big(\mathbf{\Pi}_1^*(G_i)\big)^{\prod_{j=1, j\neq i}^n |G_j|} \tag{2.2}$$

with equality if and only if $\varepsilon_{G_1}(u_1) = \varepsilon_{G_2}(u_2) = \cdots = \varepsilon_{G_n}(u_n)$ for any vertex $(u_1, u_2, \ldots, u_n)$ of $\square_{i=1}^n G_i$.

Proof. Note that G_i $(i = 1, 2, \ldots, n)$ does not contain any well-connected vertex, then for any vertex u_i of G_i, we have $\varepsilon_{G_i}(u_i) > 1$, $i = 1, 2, \ldots, n$. So, we can obtain that $\sum_{i=1}^n \varepsilon_{G_i}(u_i) \leq \prod_{i=1}^n \varepsilon_{G_i}(u_i)$. By Lemma 2.1 and the same discussing method in Theorem 2.2, it is easy to deduce the inequality (2.2) holds. □

Similarly, we can also obtain the lower and upper bounds of $\mathbf{\Pi}_2^*$ index of Cartesian product graphs.

Theorem 2.4. *Let $G_1, G_2, \ldots, G_n$ be n nontrivial graphs. Then*

$$\begin{aligned}\mathbf{\Pi}_2^*(\square_{i=1}^n G_i) &\geq n^{2\sum_{i=1}^n |E(G_i)|(\prod_{j=1,j\neq i}^n |G_j|)}\\ &\times \prod_{i=1}^n \Big[\prod_{j=1,j\neq i}^n \big(\mathbf{\Pi}_1^*(G_j)\big)^{|E(G_i)|\prod_{k=1,k\neq i,j}^n |G_k|} \times \big(\mathbf{\Pi}_2^*(G_i)\big)^{\prod_{j=1,j\neq i}^n |G_j|}\Big]^{\frac{1}{n}}\end{aligned}$$

with equality if and only if $\varepsilon_{G_1}(u_1) = \varepsilon_{G_2}(u_2) = \cdots = \varepsilon_{G_n}(u_n)$ for any vertex $(u_1, u_2, \ldots, u_n)$ in $\square_{i=1}^n G_i$. If $n = 2$, set $\prod_{k=1,k\neq i,j}^n |G_k| = 1$.

Theorem 2.5. *Let $G_i (i = 1, 2, \ldots, n)$ does not contain any well-connected vertex. Then*

$$\mathbf{\Pi}_2^*(\square_{i=1}^n G_i) \leq \prod_{i=1}^n \Big[\prod_{j=1,j\neq i}^n \big(\mathbf{\Pi}_1^*(G_j)\big)^{|E(G_i)|\prod_{k=1,k\neq i,j}^n |G_k|} \times \big(\mathbf{\Pi}_2^*(G_i)\big)^{\prod_{j=1,j\neq i}^n |G_j|}\Big]$$

with equality if and only if $\varepsilon_{G_1}(u_1) = \varepsilon_{G_2}(u_2) = \cdots = \varepsilon_{G_n}(u_n)$ for any vertex $(u_1, u_2, \ldots, u_n)$ of $\square_{i=1}^n G_i$. If $n = 2$, set $\prod_{k=1,k\neq i,j}^n |G_k| = 1$.

3. $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of join graphs

The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. The definition generalizes to the case of $n \geq 3$ graphs in a straightforward manner.

In order to finish calculations of the first and second multiplicative Zagreb eccentricity indices of join of graphs, we first present two key Lemmas as bellow.

Lemma 3.1. [4, 7] *Let $G_1, G_2, \ldots, G_n$ be n graphs. Then*

$$\begin{aligned}|E(G_1 + G_2 + \cdots + G_n)| &= \sum_{i=1}^n [|E(G_i)| + \sum_{j=1,i<j}^n |G_i||G_j|],\\ \deg_{G_1+G_2+\cdots+G_n}(v) &= \deg_{G_i}(v) + \sum_{j=1,j\neq i}^n |G_j|.\end{aligned}$$

Lemma 3.2. *For any vertex $v \in V(G_i)(1 \leq i \leq n) \subset \cup_{i=1}^n V(G_i) = V(G_1 + G_2 + \cdots + G_n)$, we have*

$$\varepsilon_{G_1+G_2+\cdots+G_n}(v) = \begin{cases} 1, & \text{if } v \in W_i;\\ 2, & \text{if } v \in V(G_i) \setminus W_i.\end{cases}$$

Theorem 3.3. *Let $G = G_1 + G_2 + \cdots + G_n$. $0 \leq |W_i| \leq |G_i|, i = 1, 2, \ldots, n$. Then*

$$\begin{aligned}\mathbf{\Pi}_1^*(G) &= 4^{\sum_{i=1}^n (|G_i|-|W_i|)},\\ \mathbf{\Pi}_2^*(G) &= 2^{\sum_{i=1}^n [2|E(G_i)|-|W_i|(|G_i|-1)+(|G_i|-|W_i|)\sum_{j=1,j\neq i}^n |G_j|]}.\end{aligned}$$

Proof. By the definition of Π_1^* index, Lemmas 3.1 and 3.2, we have

$$\mathbf{\Pi}_1^*(G) = \prod_{v\in V(G)} \varepsilon_G^2(v) = \prod_{v\in \cup_{i=1}^n W_i} 1^2 \times \prod_{v\in V(G)\setminus \cup_{i=1}^n W_i} 2^2 = 4^{|V(G)\setminus \cup_{i=1}^n W_i|} = 4^{\sum_{i=1}^n (|G_i|-|W_i|)}.$$

$$\begin{aligned}
\mathbf{\Pi}_2^*(G) &= \prod_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v) = \prod_{v\in V(G)} (\varepsilon_G(v))^{\deg_G(v)} \\
&= \prod_{i=1}^{n} [1^{\sum_{v\in W_i}\deg_G(v)} \times 2^{\sum_{v\in V(G_i)\setminus W_i}\deg_G(v)}] \\
&= 2^{\sum_{i=1}^{n}\sum_{v\in V(G_i)\setminus W_i}\deg_G(v)} \\
&= 2^{\sum_{i=1}^{n}\sum_{v\in V(G_i)\setminus W_i}[\deg_{G_i}(v)+\sum_{j=1,j\neq i}^{n}|G_j|]} \\
&= 2^{\sum_{i=1}^{n}[\sum_{v\in V(G_i)\setminus W_i}\deg_{G_i}(v)+\sum_{v\in V(G_i)\setminus W_i}\sum_{j=1,j\neq i}^{n}|G_j|]} \\
&= 2^{\sum_{i=1}^{n}[2|E(G_i)|-|W_i|(|G_i|-1)+(|G_i|-|W_i|)\sum_{j=1,j\neq i}^{n}|G_j|]}.
\end{aligned}$$

This completes the proof. □

Corollary 3.4. *Let* $nG = \underbrace{G+\cdots+G}_{\text{n times}}$ *and the set of well-connected vertices of* G *is* W. *Then*

$$\begin{aligned}
\mathbf{\Pi}_1^*(nG) &= 4^{n(|G|-|W|)}, \\
\mathbf{\Pi}_2^*(nG) &= 2^{[2n|E(G)|+2\binom{n}{2}|G|^2+n|W|-n^2|G||W|)]}.
\end{aligned}$$

Corollary 3.5. *If none of* G_i, $i=1,2,\ldots,n$ *contains well-connected vertices, then*

$$\begin{aligned}
\mathbf{\Pi}_1^*(G_1+G_2+\cdots+G_n) &= 4^{\sum_{i=1}^{n}|G_i|}, \\
\mathbf{\Pi}_2^*(G_1+G_2+\cdots+G_n) &= 4^{\sum_{i=1}^{n}[|E(G_i)|+\sum_{j=1,i<j}^{n}|G_i||G_j|]}.
\end{aligned}$$

4. $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of disjunction and symmetric difference of two graphs

The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1)\times V(G_2)$ and edge set $E(G_1\vee G_2) = \{(u_1,u_2)(v_1,v_2) | u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2)\}$.

Lemma 4.1. [4, 7] *Let* G_1 *and* G_2 *be two graphs, then*

$$\begin{aligned}
|E(G_1\vee G_2)| &= |E(G_1)||G_2|^2 + |E(G_2)||G_1|^2 - 2|E(G_1)||E(G_2)|, \\
\deg_{G_1\vee G_2}((u,v)) &= |G_2|\deg_{G_1}(u) + |G_1|\deg_{G_2}(v) - \deg_{G_1}(u)\deg_{G_2}(v).
\end{aligned}$$

The following lemma is obvious by means of the definition of disjunction of two graphs.

Lemma 4.2. *Let* G_1 *and* G_2 *be two graphs,* W_i *is the set of well-connected vertices of* G_i *and* $0 \le |W_i| \le |V(G_i)|$, $i=1,2$. *Then*

$$\varepsilon_{G_1\vee G_2}((u,v)) = \begin{cases} 1, & \text{if } u\in W_1 \text{ and } v \in W_2; \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Let* G_1 *and* G_2 *be two graphs. Then*

$$\begin{aligned}
\mathbf{\Pi}_1^*(G_1\vee G_2) &= 4^{(|G_1||G_2|-|W_1||W_2|)}, \\
\mathbf{\Pi}_2^*(G_1\vee G_2) &= 2^{[2(|E(G_1)||G_2|^2+|E(G_2)||G_1|^2-2|E(G_1)||E(G_2)|)-|W_2||G_2|s(W_1)-|W_1||G_1|s(W_2)+s(W_1)s(W_2)]},
\end{aligned}$$

where $s(W_i)=\sum_{v\in W_i}\deg_{G_i}(v), i=1,2.$

Proof. From the definitions of multiplicative Zagreb eccentricity indices $\mathbf{\Pi}_1^*$, $\mathbf{\Pi}_2^*$ and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\mathbf{\Pi}_1^*(G_1\vee G_2) &= \prod_{(u,v)\in V(G_1\vee G_2)}\varepsilon^2_{G_1\vee G_2}((u,v)) = \prod_{(u,v)\in V(G_1\vee G_2)\setminus(W_1\times W_2)} 2^2\\ &= 4^{|V(G_1\vee G_2)\setminus(W_1\times W_2)|} = 4^{(|G_1||G_2|-|W_1||W_2|)},\end{aligned}$$

$$\begin{aligned}\mathbf{\Pi}_2^*(G_1\vee G_2) &= \prod_{(u,v)\in V(G_1\vee G_2)}(\varepsilon_{G_1\vee G_2}((u,v)))^{\deg_{G_1\vee G_2}((u,v))}\\ &= \prod_{(u,v)\in V(G_1\vee G_2)\setminus(W_1\times W_2)} 2^{\deg_{G_1\vee G_2}((u,v))}\\ &= 2^{\sum_{(u,v)\in V(G_1\vee G_2)\setminus(W_1\times W_2)}\deg_{G_1\vee G_2}((u,v))}\\ &= 2^{[\sum_{(u,v)\in V(G_1\vee G_2)}\deg_{G_1\vee G_2}((u,v))-\sum_{(u,v)\in W_1\times W_2}\deg_{G_1\vee G_2}((u,v))]}\\ &= 2^{[2|E(G_1\vee G_2)|-\sum_{u\in W_1}\sum_{v\in W_2}(|G_2|\deg_{G_1}(u)+|G_1|\deg_{G_2}(v)-\deg_{G_1}(u)\deg_{G_2}(v))]}\\ &= 2^{[2(|E(G_1)||G_2|^2+|E(G_2)||G_1|^2-2|E(G_1)||E(G_2)|)-|W_2||G_2|s(W_1)-|W_1||G_1|s(W_2)+s(W_1)s(W_2)]}.\end{aligned}$$

The proof is completed. □

Corollary 4.4. *If G_1 or G_2 does not contain any well-connected vertex, then*

$$\begin{aligned}\mathbf{\Pi}_1^*(G_1\vee G_2) &= 4^{|G_1||G_2|},\\ \mathbf{\Pi}_2^*(G_1\vee G_2) &= 4^{[|E(G_1)||G_2|^2+|E(G_2)||G_1|^2-2|E(G_1)||E(G_2)|]}.\end{aligned}$$

The symmetric difference $G_1\oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1)\times V(G_2)$ and edge set $\{(u_1,u_2)(v_1,v_2)|u_1v_1\in E(G_1)$ *or* $u_2v_2\in E(G_2)$ *but not both*$\}$.

Lemma 4.5. [4] *Let G_1 and G_2 be two graphs, then*

$$\begin{aligned}|E(G_1\oplus G_2)| &= |E(G_1)||G_2|^2+|E(G_2)||G_1|^2-4|E(G_1)||E(G_2)|,\\ \deg_{G_1\oplus G_2}((u,v)) &= |G_2|\deg_{G_1}(u)+|G_1|\deg_{G_2}(v)-2\deg_{G_1}(u)\deg_{G_2}(v).\end{aligned}$$

For any vertex $(u,v)\in V(G_1\oplus G_2)$, by the definition of symmetric difference of two graphs, we have $\varepsilon_{G_1\oplus G_2}((u,v))=2$. So, by Lemma 4.5, it is easy to obtain the following theorem.

Theorem 4.6. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned}\mathbf{\Pi}_1^*(G_1\oplus G_2) &= 4^{|G_1||G_2|},\\ \mathbf{\Pi}_2^*(G_1\oplus G_2) &= 4^{[|E(G_1)||G_2|^2+|E(G_2)||G_1|^2-4|E(G_1)||E(G_2)|]}.\end{aligned}$$

5. $\mathbf{\Pi}_1^*$ and $\mathbf{\Pi}_2^*$ indices of composition of two graphs

The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and edge set $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G_1) \text{ or } (u_1 = u_2 \text{ and } v_1v_2 \in E(G_2))\}$.

By means of the definition as above, it is easy to deduce the following lemma.

Lemma 5.1. *Let W_i be the set of well-connected vertices in graph G_i, $i = 1, 2$. Then we have*

$$\varepsilon_{G_1[G_2]}((u,v)) = \begin{cases} 1, & \text{if } u \in W_1 \text{ and } v \in W_2; \\ 2, & \text{if } u \in W_1 \text{ and } v \in V(G_2) \setminus W_2; \\ \varepsilon_{G_1}(u), & \text{otherwise.} \end{cases}$$

Lemma 5.2. [4, 7] *Let G_1 and G_2 be two graphs, then*

$$\begin{aligned} |E(G_1[G_2])| &= |E(G_1)||G_2|^2 + |E(G_2)||G_1|, \\ \deg_{G_1[G_2]}((u,v)) &= |G_2|\deg_{G_1}(u) + \deg_{G_2}(v). \end{aligned}$$

Theorem 5.3. *Let G_1 and G_2 be two graphs and W_i be the set of well-connected vertices in G_i, $i = 1, 2$. Then*

$$\begin{aligned} \mathbf{\Pi}_1^*(G_1[G_2]) &= 4^{|W_1|(|G_2|-|W_2|)} \times \big(\mathbf{\Pi}_1^*(G_1)\big)^{|G_2|}, \\ \mathbf{\Pi}_2^*(G_1[G_2]) &= 2^{|W_1|[|G_2|(|G_2|-|W_2|)(|G_1|-1)+2|E(G_2)|-|W_2|(|G_2|-1)]} \\ &\qquad \times \big(\mathbf{\Pi}_1^*(G_1)\big)^{|E(G_2)|} \times \big(\mathbf{\Pi}_2^*(G_1)\big)^{|G_2|^2}. \end{aligned}$$

Proof. By the definitions of $\mathbf{\Pi}_1^*$ index and $\mathbf{\Pi}_2^*$ index, Lemmas 5.1 and 5.2, we arrive at

$$\begin{aligned} \mathbf{\Pi}_1^*(G_1[G_2]) &= \prod_{(u,v)\in V(G_1[G_2]} \varepsilon^2_{G_1[G_2]}((u,v)) \\ &= \prod_{u\in W_1, v\in W_2} 1^2 \times \prod_{u\in W_1, v\in V(G_2)\setminus W_2} 2^2 \times \prod_{u\in V(G_1)\setminus W_1, v\in V(G_2)} \varepsilon^2_{G_1}(u) \\ &= 4^{|W_1|(|G_2|-|W_2|)} \times \Big(\prod_{u\in V(G_1)\setminus W_1} \varepsilon^2_{G_1}(u)\Big)^{|G_2|} \\ &= 4^{|W_1|(|G_2|-|W_2|)} \times \big(\mathbf{\Pi}_1^*(G_1)\big)^{|G_2|}. \end{aligned}$$

$$\begin{aligned} \mathbf{\Pi}_2^*(G_1[G_2]) &= \prod_{(u,v)\in V(G_1[G_2])} (\varepsilon_{G_1[G_2]}((u,v)))^{\deg_{G_1[G_2]}((u,v))} \\ &= \prod_{u\in W_1, v\in V(G_2)\setminus W_2} 2^{[|G_2|\deg_{G_1}(u)+\deg_{G_2}(v)]} \\ &\quad \times \prod_{u\in V(G_1)\setminus W_1, v\in V(G_2)} \big(\varepsilon_{G_1}(u)\big)^{[|G_2|\deg_{G_1}(u)+\deg_{G_2}(v)]} \end{aligned}$$

$$= 2^{\sum_{u\in W_1, v\in V(G_2)\setminus W_2}[|G_2|\deg_{G_1}(u)+\deg_{G_2}(v)]} \times \prod_{u\in V(G_1)\setminus W_1, v\in V(G_2)} \big(\varepsilon_{G_1}(u)\big)^{|G_2|\deg_{G_1}(u)}$$

$$\times \prod_{u\in V(G_1)\setminus W_1, v\in V(G_2)} \big(\varepsilon_{G_1}(u)\big)^{\deg_{G_2}(v)}$$

$$= 2^{[|G_2|(|G_2|-|W_2|)\sum_{u\in W_1}\deg_{G_1}(u)+|W_1|\sum_{v\in V(G_2)\setminus W_2}\deg_{G_2}(v)]}$$

$$\times \Big(\prod_{u\in V(G_1)\setminus W_1} (\varepsilon_{G_1}(u))^{\deg_{G_1}(u)}\Big)^{|G_2|^2} \times \Big(\prod_{u\in V(G_1)\setminus W_1} \varepsilon^2_{G_1}(u)\Big)^{\frac{1}{2}\sum_{v\in V(G_2)}\deg_{G_2}(v)}$$

$$= 2^{|W_1|[|G_2|(|G_2|-|W_2|)(|G_1|-1)+2|E(G_2)|-|W_2|(|G_2|-1)]} \times \big(\mathbf{\Pi}^*_2(G_1)\big)^{|G_2|^2} \times \big(\mathbf{\Pi}^*_1(G_1)\big)^{|E(G_2)|}.$$

This completes the proof. □

Corollary 5.4. *If $W_1 = \emptyset$ or $W_2 = V(G_2)$, i.e., G_1 does not contain any well-connected vertex or G_2 is a complete graph. Then*

$$\mathbf{\Pi}^*_1(G_1[G_2]) = \big(\mathbf{\Pi}^*_1(G_1)\big)^{|G_2|} \text{ and } \mathbf{\Pi}^*_2(G_1[G_2]) = \big(\mathbf{\Pi}^*_1(G_1)\big)^{|E(G_2)|} \times \big(\mathbf{\Pi}^*_2(G_1)\big)^{|G_2|^2}.$$

Acknowledgments

The authors wish to thank the referee for helpful comments and suggestions. This work is supported by the National Natural Science Foundation of China (No.11271006), the Scientific Research Programs of the Higher Education Institution of Xinjiang (XJEDU2012I37, XJEDU2012I38) and the Science Foundation of Changji University (2012YJYB003).

References

[1] K. C. Das, D. W. Lee and A. Graovac, Some properties of the Zagreb eccentricity indices, *Ars Math. Contemp.*, **6** (2013) 117–125.
[2] N. De, On multiplicative Zagreb eccenticity indices, *South Asian J. Math.*, **2** no. 6 (2012) 570–577.
[3] Z. Du, B. Zhou and N. Trinajstić, Extremal properties of the Zagreb eccentricity indices, *Croat. Chem. Acta*, **85** no. 3 (2012) 359–362.
[4] M. Ghorbani and M. A. Hosseinzadeh, A new version of Zagreb indices, *Filomat*, **26** (2012) 93–100.
[5] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals, Total π electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
[6] I. Gutman, Multipicative Zagreb indices of trees, *Bull. Internat. Math. Virt. Inst.*, **1** (2011) 13–19.
[7] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.*, **157** (2009) 804–811.
[8] R. Todeschini and V. Consonni, New local vertex invariants and molecular descriptors based on functions of the vertex degree, *MATCH Commun. Math. Comput. Chem.*, **64** (2010) 359–372.
[9] D. Vukičević and A. Graovac, Note on the comparison of the first and second normalized Zagreb eccentricity indices, *Acta Chim. Slov.* **57** (2010) 524–528.
[10] D. B. West, *Introduction to Graph Theory*, second ed., Prentice Hall, Inc., Upper Saddle River, NJ, 1996.
[11] R. Xing, B. Zhou and N. Trinajstić, On Zagreb eccentricity indices, *Croat. Chem. Acta*, **84** no. 4 (2011) 493–497.

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