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RANDIĆ INCIDENCE ENERGY OF GRAPHS

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Abstract. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Similar to the Randić matrix, here we introduce the Randić incidence matrix of a graph G, denoted by $I_R(G)$, which is defined as the $n \times m$ matrix whose (i, j)-entry is $(d_i)^{-\frac{1}{2}}$ if v_i is incident to e_j and 0 otherwise. Naturally, the Randić incidence energy I_RE of G is the sum of the singular values of $I_R(G)$. We establish lower and upper bounds for the Randić incidence energy. Graphs for which these bounds are best possible are characterized. Moreover, we investigate the relation between the Randić incidence energy of a graph and that of its subgraphs. Also we give a sharp upper bound for the Randić incidence energy of a bipartite graph and determine the trees with the maximum Randić incidence energy among all n-vertex trees. As a result, some results are very different from those for incidence energy.

1. **Introduction**

In this paper we are concerned with simple finite graphs. Undefined notation and terminology can be found in [1]. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and let d_i be the degree of vertex v_i, $i = 1, 2, \ldots, n$.

For $S \subseteq V(G)$, $G[S]$ is used to denote the subgraph of G induced by S. For a subset E' of $E(G)$, the subgraph of G obtained by deleting the edges of E' is denoted by $G - E'$. If E' consists of only one edge e, then $G - E'$ will be written as $G - e$.

The Randić index [17] of G is defined as the sum of $\frac{1}{\sqrt{d_i d_j}}$ over all edges $v_i v_j$ of G. Let $A(G)$ be the $(0, 1)$-adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The Randić matrix

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[10] $R = R(G)$ of order n can be viewed as a weighted adjacency matrix, whose (i,j)-entry is defined as

$$R_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ (d_i d_j)^{-\frac{1}{2}} & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are adjacent,} \\ 0 & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are not adjacent.} \end{cases}$$

Denote the eigenvalues of the Randić matrix $R = R(G)$ by $\rho_1, \rho_2, \ldots, \rho_n$ and label them in non-increasing order. The greatest Randić eigenvalue has been studied in [10], that is, $\rho_1 = 1$ if G possesses at least one edge. Note that in [8] we introduced the concepts of general Randić matrix and general Randić energy and deduced some results about them.

The signless Laplacian matrix [6] of G is $Q(G) = D(G) + A(G)$. This matrix has nonnegative eigenvalues $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$. If the graph G does not possess isolated vertices, the normalized signless Laplacian matrix [4] can be defined as $\mathcal{Q}(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2}$. Let $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ be eigenvalues of $\mathcal{Q}(G)$ with $\mu_1^+ \geq \mu_2^+ \geq \cdots \geq \mu_n^+$. Then, evidently,

$$\mathcal{Q}(G) = I_n + R(G).$$

Here and later I_n is denoted the unit matrix of order n. So $\mu_i^+ = 1 + \rho_i$ for $i = 1, 2, \ldots, n$. Therefore, $\mu_1^+ = 2$.

The incidence matrix $I(G)$ of G is the $n \times m$ matrix whose (i,j)-entry is 1 if v_i is incident to e_j and 0 otherwise.

The notion of the energy of a graph G was introduced by Gutman [9] in 1978 as the sum of the absolute values of the eigenvalues of $A(G)$. Its origin was from chemistry, where it is connected with the total π-electron energy of a molecule [13]. Research on graph energy is nowadays very active, various properties of graph energy may be found in [15]. The concept of graph energy was extended to any matrix by Nikiforov [16] in the following manner. Recall that singular values of a real (not necessarily square) matrix M are the square roots of the eigenvalues of the (square) matrix MM^T or M^TM and that these matrices have the same nonzero eigenvalues. The energy $E(M)$ of the matrix M is then defined [16] as the sum of its singular values.

Motivated by Nikiforov's idea, the incidence energy $IE(G)$ of a graph G was defined [14] as the sum of the singular values of the incidence matrix $I(G)$, that, in turn, are equal to the square roots of the eigenvalues of $I(G)I(G)^T$.

We use $Line(G)$ to denote the line graph of G. It is well-known [5] that for a graph G,

$$I(G)I(G)^T = D(G) + A(G) = Q(G),$$

and

$$I(G)^T I(G) = 2I_m + A(Line(G)).$$

Some basic properties of incidence energy were established in [11, 14]. Many lower and upper bounds on this quantity were found; for details see [2, 7, 12].

Similar to the Randić matrix, in this paper we define an $n \times m$ matrix whose (i,j)-entry is $(d_i)^{-\frac{1}{2}}$ if v_i is incident to e_j and 0 otherwise, and call it the *Randić incidence matrix* of G and denote it by $I_R(G)$. Obviously, $I_R(G) = D^{-\frac{1}{2}} I(G)$ if G does not have isolated vertices.

Let U be the set of isolated vertices of G, and $W = V - U$. Set $r = |W|$ $(r \leq n)$. From the definition, we can easily get that

$$I_R(G)I_R(G)^T = \begin{pmatrix} I_r + R(G[W]) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{Q}(G[W]) & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

We can consider the Randić incidence matrix as a weighted incidence matrix. It is a natural generalization of the incidence matrix. Let $\sigma_1(G), \sigma_2(G), \ldots, \sigma_n(G)$ be the singular values of the Randić incidence matrix of a graph G. Now we define $I_RE(G) := \sum_{i=1}^{n} \sigma_i(G)$, which is called the *Randić incidence energy* of G.

Also if the graph G has components $G_1, \ldots, G_k$, such that each of those is not an isolated vertex, then $I_RE(G) = \sum_{i=1}^{k} I_RE(G_i)$. From (1), we know that

$$I_RE(G) = \sum_{i=1}^{r} \sqrt{\mu_i^+(G[W])}, \tag{2}$$

and

$$\sum_{i=1}^{n} \sigma_i^2(G) = tr(I_R(G)I_R(G)^T) = r. \tag{3}$$

In particular, if G has no isolated vertices, we have that

$$I_R(G)I_R(G)^T = \mathcal{Q}(G), \tag{4}$$

and

$$\sum_{i=1}^{n} \sigma_i^2(G) = n. \tag{5}$$

On the other hand, let us consider the $m \times m$ matrix $I_R(G)^T I_R(G)$. It is easy to see that its (i,j)-entry is as follows.

$$[I_R(G)^T I_R(G)]_{i,j} = \begin{cases} \frac{1}{d_k} + \frac{1}{d_\ell} & \text{if } i = j,\ e_i = \{v_k, v_\ell\}, \\ \frac{1}{d_k} & \text{if } i \neq j,\ e_i \text{ and } e_j \text{ have a common vertex } v_k \text{ in } G, \\ 0 & \text{if } i \neq j,\ e_i \text{ and } e_j \text{ do not have a common vertex in } G. \end{cases}$$

Although it looks quite different from $I^T(G)I(G)$, it has many similar properties. For examples, the sum of each column, respectively, each row, of the matrix $I_R(G)^T I_R(G)$ is 2, and $tr(I_R(G)^T I_R(G)) = n$. Therefore, 2 is its an eigenvalue with eigenvector $(1, 1, \ldots, 1)^T$. Particularly, if G is a d-regular graph, then we have

$$I_R(G)^T I_R(G) = \frac{2}{d} I_m + \frac{1}{d} A(Line(G)).$$

Let $\lambda_1(Line(G)), \cdots, \lambda_m(Line(G))$ denote the adjacency eigenvalues of $Line(G)$, clearly,

$$I_RE(G) = \sum_{i=1}^{n} \sigma_i(G) = \sum_{i=1}^{m} \sqrt{\frac{2}{d} + \frac{1}{d}\lambda_i(Line(G))}.$$

Now we give an example to calculate the Randić incidence matrix and Randić incidence energy of a special class of graphs.

Example: Consider the star S_n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_{n-1}\}$ where $e_i = v_i v_n$ for $i = 1, 2, \ldots, n-1$. Then

$$I_R(S_n) = \begin{pmatrix} I_{n-1} \\ \alpha \end{pmatrix},$$

where $\alpha = (\frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-1}}, \ldots, \frac{1}{\sqrt{n-1}})$.

$$I_R(S_n)I_R(S_n)^T = \begin{pmatrix} I_{n-1} & \alpha^T \\ \alpha & 1 \end{pmatrix} = I_n + \begin{pmatrix} O_{n-1} & \alpha^T \\ \alpha & 0 \end{pmatrix} = I_n + R(S_n).$$

By calculations, we have $\mu_1^+(S_n) = 2$, $\mu_2^+(S_n) = \cdots = \mu_{n-1}^+(S_n) = 1$ and $\mu_n^+(S_n) = 0$, and then $I_RE(S_n) = n - 2 + \sqrt{2}$.

In this paper, we establish some lower and upper bounds for the Randić incidence energy of a graph. Graphs for which these bounds are best possible are characterized. Moreover, we investigate the relation between the Randić incidence energy of a graph and that of its subgraphs. That property is analogues to incidence energy. Also we give a sharp upper bound for the Randić incidence energy of a bipartite graph and describe the trees which have the maximum Randić incidence energy among all n-vertex trees. It is interesting that the extremal trees attain the maximum Randić incidence energy are quite different from the trees which have the maximum incidence energy.

2. Upper and lower bounds

Theorem 2.1. *Let G be a graph of order n, and contains no isolated vertices. Then*

$$I_RE(G) \geq \sqrt{n}, \tag{6}$$

the equality holds if and only if $G \cong K_2$.

Proof. It is obvious that $\sum_{i=1}^{n} \sigma_i \geq \sqrt{\sum_{i=1}^{n} \sigma_i^2}$ and the equality holds if and only if at most one of the σ_i is non-zero. From (5), we know that $\sum_{i=1}^{n} \sigma_i^2(G) = n$. Therefore

$$I_RE(G) = \sum_{i=1}^{n} \sigma_i \geq \sqrt{\sum_{i=1}^{n} \sigma_i^2} = \sqrt{n}.$$

The equality holds if and only if at most one of the σ_i is non-zero, that is, the rank of $(I_R(G)I_R(G)^T)$ is 1. This is equivalent to $rank(I_R(G)) = 1$ since $rank(I_R(G)I_R(G)^T) = rank(I_R(G))$. Therefore, each component of G must have exactly one edge, i.e., each component is isomorphic to K_2. If the graph G has more than one component, clearly, $rank(I_R(G)) > 1$, a contradiction, and hence $G \cong K_2$. □

Remark 2.1: Note that in [14] there is a similar lower bound for incidence energy similar to (6), that is, $IE(G) \geq \sqrt{2m}$, the equality holds if and only if $m \leq 1$.

Theorem 2.2. *Let G be a graph of order n $(n \geq 2)$, and contains no isolated vertices. Then*

$$I_R E(G) \leq \sqrt{2} + \sqrt{(n-1)(n-2)}, \tag{7}$$

the equality holds if and only if $G \cong K_n$.

Proof. By applying the Cauchy-Schwarz inequality we have that

$$\sum_{i=2}^{n} \sqrt{\mu_i^+(G)} \leq \sqrt{(n-1)\sum_{i=2}^{n} \mu_i^+(G)},$$

with equality holds if and only if $\mu_2^+(G) = \mu_3^+(G) = \cdots = \mu_n^+(G) = \frac{n-2}{n-1} = 1 - \frac{1}{n-1}$. Since $\sum_{i=1}^{n} \mu_i^+(G) = n$ and $\mu_1^+(G) = 2$ if G has at least one edge, then

$$\begin{aligned} I_R E(G) &= \sum_{i=1}^{n} \sqrt{\mu_i^+(G)} = \sqrt{2} + \sum_{i=2}^{n} \sqrt{\mu_i^+(G)} \\ &\leq \sqrt{2} + \sqrt{(n-1)(n-\mu_1^+(G)} = \sqrt{2} + \sqrt{(n-1)(n-2)}. \end{aligned}$$

The equality is attained if and only if $\mu_2^+(G) = \mu_3^+(G) = \cdots = \mu_n^+(G) = 1 - \frac{1}{n-1}$. Then, $\rho_2(G) = \rho_3(G) = \cdots = \rho_n(G) = -\frac{1}{n-1}$, and therefore we have $G \cong K_n$.

Conversely, if $G \cong K_n$, we can easily check that $I_R E(G) = \sqrt{2} + \sqrt{(n-1)(n-2)}$. □

Remark 2.2: In [14] there is an upper bound for incidence energy, that is, $IE(G) \leq \sqrt{2mn}$, the equality holds if and only if $m = 0$. This result is quite different from ours.

3. Randić incidence energy of subgraphs

At the beginning of this section, we review some concepts in matrix theory.

Let A and B be complex matrices of order r and s, respectively $(r \geq s)$. We say the eigenvalues of B interlace the eigenvalues of A, if $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{r-s+i}(A)$ for $i = 1, \ldots, s$.

Lemma 3.1. [5, p.51] *If A and B are real symmetric matrices of order n and $C = A + B$, then*

$$\begin{aligned} \lambda_{i+j+1}(C) &\leq \lambda_{i+1}(A) + \lambda_{j-1}(B) \\ \lambda_{n-i-j}(C) &\geq \lambda_{n-i}(A) + \lambda_{n-j}(B) \end{aligned}$$

for $i, j = 0, \ldots, n$ and $i + j \leq n - 1$. In particular, for all integer i $(1 \leq i \leq n)$,

$$\lambda_i(C) \geq \lambda_i(A) + \lambda_n(B). \tag{8}$$

Theorem 3.2. *Let G be a graph and E' be a nonempty subset of $E(G)$. Then*

$$I_R E(G) > I_R E(G - E'). \tag{9}$$

Proof. Let H be the spanning subgraph of G such that $E(H) = E'$. The Randić incidence matrix of G can be partitioned as $I_R(G) = \begin{pmatrix} I_R(H) & I_R(G-E') \end{pmatrix}$, and so

$$I_R(G)I_R(G)^T = I_R(H)I_R(H)^T + I_R(G-E')I_R(G-E')^T.$$

Since $I_R(H)I_R(H)^T$ is positive semi-definite, by Eq.(8), $\lambda_i(I_R(G)I_R(G)^T) \geq \lambda_i(I_R(G-E')I_R(G-E')^T)$ for $i = 1, 2, \ldots, n$. It follows that $I_RE(G) \geq I_RE(G-E')$.

Moreover, $\lambda_i(I_R(G)I_R(G)^T) = \lambda_i(I_R(G-E')I_R(G-E')^T)$ for all i if the equality holds. Consequently, $tr(I_R(G)I_R(G)^T) = tr(I_R(G-E')I_R(G-E')^T)$, and it implies that $tr(I_R(H)I_R(H)^T) = 0$. Since $I_R(H)I_R(H)^T$ is positive semi-definite, $\lambda_i(I_R(H)I_R(H)^T) = 0$, $i = 1, 2, \ldots, n$. H must be an empty graph, a contradiction. □

Remark 3.1: For incidence energy, in [14] there is an analogous theorem, that is, the incidence energy of a graph is greater than that of its proper subgraphs.

According to $I_RE(K_n) = \sqrt{2} + \sqrt{(n-1)(n-2)}$, the following corollary is obvious.

Corollary 3.3. *Let G be a non-empty graph with clique number c. Then $I_RE(G) \geq \sqrt{2} + \sqrt{(c-1)(c-2)}$. In particular, if G has at least one edge then $I_RE(G) \geq \sqrt{2}$.*

When the edge subset E' consists of exactly one edge, we have the following theorem.

Theorem 3.4. *Let G be a connected graph, $e = \{uv\}$ be an edge of G. Then*

$$I_RE(G) \geq \sqrt{\frac{1}{d(u)} + \frac{1}{d(v)} + [I_RE(G-e)]^2}, \tag{10}$$

where $d(u)$ and $d(v)$ denote the degree of u and v, respectively. Moreover, the equality holds if and only if $G \cong K_2$.

Proof. It is easy to see that the Randić incidence matrix of G can be represented in the form of

$$I_R(G) = (I_R(G-e) \quad \beta),$$

where β is a vector of size n whose first two components are $\frac{1}{\sqrt{d(u)}}$ and $\frac{1}{\sqrt{d(v)}}$, the other components are 0. Therefore,

$$I_R(G)I_R(G)^T = I_R(G-e)I_R(G-e)^T + \begin{pmatrix} J & O \\ 0 & 0 \end{pmatrix},$$

where $J = \begin{pmatrix} \frac{1}{d(u)} & \frac{1}{\sqrt{d(u)d(v)}} \\ \frac{1}{\sqrt{d(u)d(v)}} & \frac{1}{d(v)} \end{pmatrix}$. Hence,

$$tr(I_R(G)I_R(G)^T) = \frac{1}{d(u)} + \frac{1}{d(v)} + tr(I_R(G-e)I_R(G-e)^T), \tag{11}$$

By (8), we have $\sigma_i(G) \geq \sigma_i(G-e)$ for all $i = 1, \ldots, n$.

$$\begin{aligned}
[I_RE(G)]^2 &= \sum_i {\sigma_i}^2(G) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&= tr(I_R(G)I_R(G)^T) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&= \frac{1}{d(u)} + \frac{1}{d(v)} + tr(I_R(G-e)I_R(G-e)^T) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \qquad (by\ (11)) \\
&= \frac{1}{d(u)} + \frac{1}{d(v)} + \sum_i {\sigma_i}^2(G-e) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&\geq \frac{1}{d(u)} + \frac{1}{d(v)} + \sum_i {\sigma_i}^2(G-e) + 2\sum_{i<j} \sigma_i(G-e)\sigma_j(G-e) \\
&= \frac{1}{d(u)} + \frac{1}{d(v)} + [I_RE(G-e)]^2.
\end{aligned}$$

If $I_R(G)$ has at least two non-zero singular values, since $I_RE(G) > I_RE(G-e)$, then $\sigma_k(G) > \sigma_k(G-e)$ for some $2 \leq k \leq n$. Thus,

$$\sum_{i<j} \sigma_i(G)\sigma_j(G) = \sigma_1(G)\sigma_k(G) + \sum_{i<j,(i,j)\neq(1,k)} \sigma_i(G)\sigma_j(G).$$

Since $\sigma_1(G)\sigma_k(G) > 0$, we have that

$$\sum_{i<j} \sigma_i(G)\sigma_j(G) > \sum_{i<j} \sigma_i(G-e)\sigma_j(G-e).$$

Thus, if the Randić incidence matrix of the graph G has more than one non-zero singular value, the equality in (10) does not occur. Since $rank(I_R(G)I_R(G)^T) = rank(I_R(G))$, then in the equality case, $rank(I_R(G))$ must be equal to 1. On the other hand, if the graph G has more than one edge, $rank(I_R(G)) > 1$. Therefore, the equality in (10) holds if and only if $G = K_2$. □

Remak 3.2: Here we point out that if we set $d(u) = d(v) = 1$ in (10), it is just the relation between $IE(G)$ and $IE(G-e)$ given in [14].

4. **Upper bound for bipartite graphs**

Theorem 4.1. *Let G be a bipartite graph of order n without isolated vertices. Then*

$$I_RE(G) \leq n - 2 + \sqrt{2}, \tag{12}$$

the equality holds if and only if G is a complete bipartite graph.

Proof. Let $Q(G)$ be the signless Laplacian matrix of G, and $q_1 \geq q_2 \geq \cdots \geq q_n$ be the signless Laplacian spectrum of G. It is well known that if G is bipartite, $q_n = 0$. Since $\mathcal{Q}(G) = D(G)^{-\frac{1}{2}}Q(G)D(G)^{-\frac{1}{2}}$, we get that $\mu_n^+ = 0$. Hence, $\sum_{i=1}^{n-1} \mu_i^+ = n$. Since G has at least one edge, $\mu_1^+ = 2$. By the Cauchy-Schwarz inequality, we have

$$\sum_{i=2}^{n-1}\sqrt{\mu_i^+}\le\sqrt{(n-2)(n-\mu_1^+)}=\sqrt{(n-2)(n-2)}=n-2.$$

So, we have

$$I_RE(G)=\sqrt{\mu_1^+}+\sum_{i=2}^{n-1}\sqrt{\mu_i^+}\le n-2+\sqrt{2},$$

the equality holds if $\mu_2^+=\cdots=\mu_{n-1}^+=\frac{n-2}{n-2}=1$, and $\mu_1^+=2$, $\mu_n^+=0$. This implies that the Randić eigenvalues ρ_i of G satisfies $\rho_2=\cdots=\rho_{n-1}=0$, $\rho_1=1$ and $\rho_n=-1$, i.e., $\rho_1=1$ is the only positive Randić eigenvalue of G. From Theorem 2.4 in [10], G is a complete multipartite graph. Since G is a bipartite graph, we derive that G is a complete bipartite graph. Conversely, let $G=(X,Y)$ be a complete bipartite graph with two vertex classes X and Y, where $|X|=x$, $|Y|=y$ and $x+y=n$. As is known in [5], the adjacency eigenvalues of G are $\sqrt{xy}$, $-\sqrt{xy}$, 0 ($n-2$ times). It is easy to get that $\mathcal{Q}(G)=I_n+\frac{1}{\sqrt{xy}}A(G)$, where $A(G)$ is the adjacency matrix of G. So, we have $\mu_1^+=2$, $\mu_n^+=0$ and $\mu_2^+=\cdots=\mu_{n-1}^+=1$. Thus, $I_RE(G)=n-2+\sqrt{2}$. □

Because trees are bipartite graphs, we characterized the unique tree with maximum Randić incidence energy.

Theorem 4.2. *Among all trees with n vertices, the star S_n is the unique graph with maximum Randić incidence energy.*

Proof. Let G be a tree with n vertices. Suppose that G is not a complete bipartite graph. Then G is a spanning subgraph of some complete bipartite graph $K_{s,t}$, where $s+t=n$. So, $I_RE(G)<I_RE(K_{s,t})$. From Theorem 4.1, we know that $I_RE(K_{1,n-1})=I_RE(K_{s,t})=n-2+\sqrt{2}$. So, $I_RE(G)<I_RE(K_{1,n-1})$.

If G is a complete bipartite graph $K_{x,y}$, then one of x and y must be equal to 1; otherwise, there exists a cycle in G, a contradiction. So G must be the star $K_{1,n-1}$. Hence, the star $K_{1,n-1}$ is the unique graph with maximum Randić incidence energy among all n-vertex trees. □

Remak 4.1: The same problem has been studied for incidence energy in [11], where the authors proved that for any n-vertex tree T, $IE(S_n)\le IE(T)\le IE(P_n)$, where P_n denotes the path on n vertices. But, our result says that $I_RE(T)\le I_RE(S_n)$. However, we do not know if $I_RE(P_n)\le I_RE(T)$ holds. For Randić index, it was showed in [3, 18] that among trees with n vertices, the star S_n has the minimum Randić index and the path P_n attains the maximum Randić index.

To end this paper, we point out that one can generalize the concepts Randić incidence matrix and Randić incidence energy. Similar to those in [8], define the general Randić incidence matrix of a graph G as an $n\times m$ matrix whose (i,j)-entry is $(d_i)^\alpha$ if v_i is incident to e_j and 0 otherwise, where $\alpha\ne 0$ is a fixed real number. Also, define the general Randić incidence energy as the sum of the singular values of the general Randić incidence matrix of a graph G. It could be interesting to further study these generalized concepts and get some unexpected results.

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