



SKEW-SPECTRA AND SKEW ENERGY OF VARIOUS PRODUCTS OF GRAPHS

X. LI AND H. LIAN*

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ABSTRACT. Given a graph G , let G^σ be an oriented graph of G with the orientation σ and skew-adjacency matrix $S(G^\sigma)$. Then the spectrum of $S(G^\sigma)$ consisting of all the eigenvalues of $S(G^\sigma)$ is called the skew-spectrum of G^σ , denoted by $Sp(G^\sigma)$. The skew energy of the oriented graph G^σ , denoted by $\mathcal{E}_S(G^\sigma)$, is defined as the sum of the norms of all the eigenvalues of $S(G^\sigma)$. In this paper, we give orientations of the Kronecker product $H \otimes G$ and the strong product $H * G$ of H and G where H is a bipartite graph and G is an arbitrary graph. Then we determine the skew-spectra of the resultant oriented graphs. As applications, we construct new families of oriented graphs with optimum skew energy. Moreover, we consider the skew energy of the orientation of the lexicographic product $H[G]$ of a bipartite graph H and a graph G .

1. Introduction

Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, and let G^σ be an oriented graph of G with the orientation σ , which assigns to each edge of G a direction so that the induced graph G^σ becomes an oriented graph or a directed graph. Then G is called the underlying graph of G^σ . The *skew-adjacency matrix* of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $\langle v_i, v_j \rangle$ is an arc of G^σ , otherwise $s_{ij} = s_{ji} = 0$. It is easy to see that $S(G^\sigma)$ is a skew-symmetric matrix, and thus all its eigenvalues are purely imaginary numbers or 0, which form the spectrum of $S(G^\sigma)$ and are said to be the *skew-spectrum* $Sp(G^\sigma)$ of G^σ .

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*Corresponding author.

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The concept of the energy of a simple undirected graph was introduced by Gutman in [6]. Then Adiga, Balakrishnan and So in [1] generalized the energy of an undirected graph to the skew energy of an oriented graph. Formally, the *skew energy* of an oriented graph G^σ is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$, denoted by $\mathcal{E}_S(G^\sigma)$. Most of the results on the skew energy are collected in our recent survey [7], among which the problem about the optimum skew energy has been paid more attention.

In [1], Adiga, Balakrishnan and So derived that for any oriented graph G^σ with order n and maximum degree Δ , $\mathcal{E}_S(G^\sigma) \leq n\sqrt{\Delta}$. They also showed that the equality holds if and only if $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$, which implies that G^σ is Δ -regular. Among all oriented graphs with order n and maximum degree Δ , the skew energy $n\sqrt{\Delta}$ is called the optimum skew energy. Naturally, they proposed the following problem:

Problem 1.1. *Which k -regular graphs on n vertices have orientations G^σ with $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$, or equivalently, $S(G^\sigma)^T S(G^\sigma) = kI_n$?*

For $k = 1, 2, 3, 4$, all k -regular graphs which have orientations G^σ with $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$ were characterized, see [1, 5, 3]. Other families of oriented regular graphs with the optimum skew energy were also obtained. Tian in [9] gave the orientation of the hypercube Q_k such that the resultant oriented graph has optimum skew energy. In [1], a family of oriented graphs with optimum skew energy was constructed by considering the Kronecker product of graphs. To be specific, let $G_1^{\sigma_1}, G_2^{\sigma_2}, G_3^{\sigma_3}$ be the oriented graphs of order n_1, n_2, n_3 with skew-adjacency matrices S_1, S_2, S_3 , respectively. Then the Kronecker product matrix $S_1 \otimes S_2 \otimes S_3$ is also skew-symmetric and is in fact the skew-adjacency matrix of an oriented graph of the Kronecker product $G_1 \otimes G_2 \otimes G_3$. Denote the corresponding oriented graph by $G_1^{\sigma_1} \otimes G_2^{\sigma_2} \otimes G_3^{\sigma_3}$. The following result was obtained.

Theorem 1.2. [1] *Let $G_1^{\sigma_1}, G_2^{\sigma_2}, G_3^{\sigma_3}$ be the oriented regular graphs of order n_1, n_2, n_3 with optimum skew energies $n_1\sqrt{k_1}, n_2\sqrt{k_2}, n_3\sqrt{k_3}$, respectively. Denote by S_1, S_2 and S_3 the skew-adjacency matrices of $G_1^{\sigma_1}, G_2^{\sigma_2}$ and $G_3^{\sigma_3}$, respectively. Then the oriented graph $G_1^{\sigma_1} \otimes G_2^{\sigma_2} \otimes G_3^{\sigma_3}$ has optimum skew energy $n_1 n_2 n_3 \sqrt{k_1 k_2 k_3}$.*

It should be noted that the above Kronecker product of oriented graphs is naturally defined, but the product requires 3 or an odd number of oriented graphs.

Moreover, Cui and Hou in [4] gave an orientation $(P_m \square G)^\circ$ of the Cartesian product $P_m \square G$, where P_m is a path of order m and G is an arbitrary graph. They computed the skew-spectra of $(P_m \square G)^\circ$, and by applying this result they constructed a family of oriented graphs with optimum skew energy. Then we in [2] extended their results to the oriented graph $(H \square G)^\circ$ where H is an arbitrary bipartite graph, and thus a larger family of oriented graphs with optimum skew energy was obtained.

Theorem 1.3. [2] *Let H^τ be an oriented ℓ -regular bipartite graph on m vertices with optimum skew energy $\mathcal{E}_S(H^\tau) = m\sqrt{\ell}$ and G^σ be an oriented k -regular graph on n vertices with optimum skew energy $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$. Then the oriented graph $(H^\tau \square G^\sigma)^\circ$ of $H \square G$ has the optimum skew energy $\mathcal{E}_S((H^\tau \square G^\sigma)^\circ) = mn\sqrt{\ell + k}$.*

In this paper, we consider other products of graphs, including the Kronecker product $H \otimes G$, the strong product $H * G$ and the lexicographic product $H[G]$, where H is a bipartite graph and G is an arbitrary graph. In Subsection 2.1, We first give an orientation of $H \otimes G$, and then determine the skew-spectra of the resultant oriented graph. As an application, we construct a new family of oriented graphs with optimum skew energy. Subsection 2.2 is used to orient the graph $H * G$, determine the skew-spectra of the resultant oriented graph and construct another new family of oriented graphs with optimum skew energy. Finally we consider the skew energy of the orientation of the lexicographic product $H[G]$ of H and G in Subsection 2.3.

In the sequel of this paper, it will be seen that there is no limitation of the number of oriented graphs in our Kronecker product, and the oriented graphs that we will construct have smaller order than the previous results under the same regularity.

2. Main results

We first recall some definitions. Let H be a graph of order m and G be a graph of order n . The Cartesian product $H \square G$ of H and G has vertex set $V(H) \times V(G)$, where (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and v_1 is adjacent to v_2 in G , or u_1 is adjacent to u_2 in H and $v_1 = v_2$. The Kronecker product $H \otimes G$ of H and G is a graph with vertex set $V(H) \times V(G)$ and where (u_1, v_1) and (u_2, v_2) are adjacent if u_1 is adjacent to u_2 in H and v_1 is adjacent to v_2 in G . The strong product $H * G$ of H and G is a graph with vertex set $V(H) \times V(G)$; two distinct pairs (u_1, v_1) and (u_2, v_2) are adjacent in $H * G$ if u_1 is equal or adjacent to u_2 , and v_1 is equal or adjacent to v_2 . The lexicographic product $H[G]$ of H and G has vertex set $V(H) \times V(G)$ where (u_1, v_1) is adjacent to (u_2, v_2) if and only if u_1 is adjacent to u_2 in H , or $u_1 = u_2$ and v_1 is adjacent to v_2 in G .

It can be verified that the Cartesian product $H \square G$, the Kronecker product $H \otimes G$, the strong product $H * G$ are commutative, that is, $H \square G = G \square H$, $H \otimes G = G \otimes H$ and $H * G = G * H$. But the lexicographic product $H[G]$ may not be the same as $G[H]$. Moreover, the two graphs $H \square G$ and $H \otimes G$ are edge-disjoint and $E(H * G) = E(H \square G) \cup E(H \otimes G)$. Finally, we point out that if H is a bipartite graph, then $H \otimes G$ is also bipartite.

In what follows, we always assume that H is a bipartite graph on m vertices with bipartite (X, Y) where $|X| = m_1$ and $|Y| = m_2$ and G is a graph on n vertices. Let H^τ be an arbitrary oriented graph of H and G^σ be an arbitrary oriented graph of G . Let S_1 and S_2 be the skew-adjacency matrices of H^τ and G^σ , respectively. Giving the labeling of the vertices of H such that the vertices of X are labeled first. Then the skew-adjacency matrix S_1 can be formulated as $\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$, where A is an $m_1 \times m_2$ matrix and $m_1 + m_2 = m$. Let $S'_1 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$. Note that S_1 is skew-symmetric and S'_1 is symmetric. It is easy to see that $S_1 S_1^T = S'_1 (S'_1)^T$, and thus S_1 and S_2 have the same singular values.

2.1. The orientation of $H \otimes G$. We first give an orientation of $H \otimes G$. For any two adjacent vertices (u_1, v_1) and (u_2, v_2) , u_1 and u_2 must be in different parts of the bipartition of vertices of H and assume

that $u_1 \in X$. Then there is an arc from (u_1, v_1) to (u_2, v_2) if $\langle u_1, u_2 \rangle$ is an arc of H and $\langle v_1, v_2 \rangle$ is an arc of G , or $\langle u_2, u_1 \rangle$ is an arc of H and $\langle v_2, v_1 \rangle$ is an arc of G ; otherwise there is an arc from (u_2, v_2) to (u_1, v_1) . Denote by $(H^\tau \otimes G^\sigma)^o$ the resultant oriented graph and by S its skew-adjacency matrix. For the skew-spectrum of $(H^\tau \otimes G^\sigma)^o$, we obtain the following result.

Theorem 2.1. *Let H^τ be an oriented bipartite graph of order m and let the skew-eigenvalues of H^τ be the non-zero values $\pm\mu_1i, \pm\mu_2i, \dots, \pm\mu_ti$ and $m - 2t$ 0's. Let G^σ be an oriented graph of order n and let the skew-eigenvalues of G^σ be the non-zero values $\pm\lambda_1i, \pm\lambda_2i, \dots, \pm\lambda_ri$ and $n - 2r$ 0's. Then the skew-eigenvalues of the oriented graph $(H^\tau \otimes G^\sigma)^o$ are $\pm\mu_j\lambda_ki$ with multiplicities 2, $j = 1, \dots, t$, $k = 1, \dots, r$, and 0 with multiplicities $mn - 4rt$.*

Proof. With suitable labeling of the vertices of $H \otimes G$, the skew-adjacency matrix S of $(H^\tau \otimes G^\sigma)^o$ can be formulated as $S = S'_1 \otimes S_2$. We first compute the singular values of S . Note that $S^T = -S'_1 \otimes S_2$. Then

$$SS^T = (S'_1 \otimes S_2)(-S'_1 \otimes S_2) = -(S'_1)^2 \otimes S_2^2.$$

It follows that the eigenvalues of SS^T are $\mu(S'_1)^2 \cdot \lambda(S_2)^2$, where $\mu(S'_1)$ is the eigenvalues of S'_1 and $\lambda(S_2)i$ is the eigenvalues of S_2 . That is to say, the eigenvalues of SS^T are $\mu(H^\tau)^2 \cdot \lambda(G^\sigma)^2$, where $\mu(H^\tau)i \in Sp(H^\tau)$ and $\lambda(G^\sigma)i \in Sp(G^\sigma)$. From this, it immediately follows what we want. The proof is now complete. \square

The above theorem can be used to yield a family of oriented graphs with optimum skew energy. The following lemma was obtained in [1].

Lemma 2.2. [1] *Let G^σ be an oriented graph of G with order n and maximum degree Δ . Then $\mathcal{E}_S(G^\sigma) \leq n\sqrt{\Delta}$, where the equality holds if and only if $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$.*

Theorem 2.3. *Let H^τ be an oriented k -regular bipartite graph of order m with optimum skew energy $m\sqrt{k}$. Let G^σ be an oriented ℓ -regular graph of order n and the optimum skew energy $n\sqrt{\ell}$. Then $(H^\tau \otimes G^\sigma)^o$ is an oriented $k\ell$ -regular bipartite graph and has the optimum skew energy $\mathcal{E}_S((H^\tau \otimes G^\sigma)^o) = mn\sqrt{k\ell}$.*

Proof. By the definition of the Kronecker product, it is easy to find that $H \otimes G$ is a $k\ell$ -regular bipartite graph with mn vertices. Let $S(H^\tau) = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$ be the skew-adjacency matrix of H^τ and $S(G^\sigma)$ be the skew-adjacency matrix of G^σ . Then by Lemma 2.2, we have $S(H^\tau)^T S(H^\tau) = kI_m$ and $S(G^\sigma)^T S(G^\sigma) = \ell I_n$. From Theorem 2.1, the skew-adjacency matrix S of $(H^\tau \otimes G^\sigma)^o$ can be written as $S = S'(H^\tau) \otimes S(G^\sigma)$, where $S'(H^\tau) = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$. Note that $S'(H^\tau)^T S'(H^\tau) = S(H^\tau)^T S(H^\tau) = kI_m$. It follows that

$$\begin{aligned} S^T S &= (S'(H^\tau) \otimes S(G^\sigma))^T (S'(H^\tau) \otimes S(G^\sigma)) \\ &= ((S'(H^\tau))^T S'(H^\tau)) \otimes (S(G^\sigma))^T (S(G^\sigma)) \\ &= kI_m \otimes \ell I_n = k\ell I_{mn}. \end{aligned}$$

By Lemma 2.2, the oriented graph $(H^\tau \otimes G^\sigma)^\circ$ has the optimum skew energy $\mathcal{E}_S((H^\tau \otimes G^\sigma)^\circ) = mn\sqrt{k\ell}$. We thus complete the proof of this theorem. \square

Let H^τ be an oriented bipartite graph with optimum skew energy. Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be any two oriented graphs with the optimum skew energies. By the above theorem, the oriented graph $(H^\tau \otimes G_1^{\sigma_1})^\circ$ is bipartite and has the optimum skew energy. Therefore, the Kronecker product $H \otimes G_1 \otimes G_2$ can be oriented as $((H^\tau \otimes G_1^{\sigma_1})^\circ \otimes G_2^{\sigma_2})^\circ$, abbreviated as $(H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2})^\circ$, which is also bipartite and has the optimum skew energy. The process is valid for any positive integral number of oriented graphs. Then the following corollary is immediately implied.

Corollary 2.4. *Let H^τ be an oriented k -regular bipartite graph of order m with optimum skew energy $m\sqrt{k}$. Let $G_i^{\sigma_i}$ be an oriented ℓ_i -regular graph of order n_i with optimum skew energy $n_i\sqrt{\ell_i}$ for $i = 1, 2, \dots, s$ and any positive integer s . Then the oriented graph $(H^\tau \otimes G_1^{\sigma_1} \otimes \dots \otimes G_s^{\sigma_s})^\circ$ has the optimum skew energy $mn_1n_2 \dots n_s\sqrt{k\ell_1\ell_2 \dots \ell_s}$.*

Remark 2.5. *In Corollary 2.4, the value s can be any positive integer. If $s = 2$, then the oriented graph $(H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2})^\circ$ has the optimum skew energy $mn_1n_2\sqrt{k\ell_1\ell_2}$. Recall the orientation in Theorem 1.2, which illustrates that the oriented graph $H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2}$ also has the optimum skew energy $mn_1n_2\sqrt{k\ell_1\ell_2}$. In fact, this two orientations are identical.*

Let $S_0 = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$ be the skew-adjacency matrix of H^τ and $S'_0 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$. Let S_1 and S_2 be the skew-adjacency matrices of $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$. Then the oriented graph $H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2}$ has the skew-adjacency matrix $\begin{pmatrix} 0 & A \otimes S_1 \otimes S_2 \\ -A^T \otimes S_1 \otimes S_2 & 0 \end{pmatrix}$. The oriented graph $(H^\tau \otimes G_1^{\sigma_1})^\circ$ has the skew-adjacency matrix

$$S'_1 \otimes S_2 = \begin{pmatrix} 0 & A \otimes S_1 \\ A^T \otimes S_1 & 0 \end{pmatrix}.$$

It follows that the skew-adjacency matrix of $(H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2})^\circ$ is

$$\begin{pmatrix} 0 & A \otimes S_1 \\ -A^T \otimes S_1 & 0 \end{pmatrix} \otimes S_2 = \begin{pmatrix} 0 & A \otimes S_1 \otimes S_2 \\ -A^T \otimes S_1 \otimes S_2 & 0 \end{pmatrix},$$

which is the same as that of $H^\tau \otimes G_1^{\sigma_1} \otimes G_2^{\sigma_2}$.

In fact, for any even s , the oriented graph obtained in Corollary 2.4 is identical to the one obtained in Theorem 1.2.

2.2. The orientation of $H * G$. Now we consider the strong product $H * G$ of a bipartite graph H and a graph G , Let H^τ be an oriented graph of H and G^σ be an oriented graph of G . Since the edge set of $H * G$ is the disjoint-union of the edge sets of $H \square G$ and $H \otimes G$, there is a natural orientation of $H * G$ if $H \square G$ and $H \otimes G$ have been given orientations.

First recall the orientation $(H^\tau \square G^\sigma)^\circ$ of the Cartesian product $H \square G$ given in [2]. For any two adjacent matrices (u_1, v_1) and (u_2, v_2) , we give it an orientation as follows. When $u_1 = u_2 \in X$, there is an arc from (u_1, v_1) to (u_2, v_2) if $\langle v_1, v_2 \rangle$ is an arc of G^σ and an arc from (u_2, v_2) to (u_1, v_1)

otherwise. When $u_1 = u_2 \in Y$, there is an arc from (u_1, v_1) to (u_2, v_2) if $\langle v_2, v_1 \rangle$ is an arc of G^σ and an arc from (u_2, v_2) to (u_1, v_1) otherwise. When $v_1 = v_2$, there is an arc from (u_1, v_1) to (u_2, v_2) if $\langle u_1, u_2 \rangle$ is an arc of H^τ and an arc from (u_2, v_2) to (u_1, v_1) otherwise. Let \bar{S} be the skew-adjacency matrix of $(H^\tau \square G^\sigma)^o$.

Now we give an orientation of $H * G$ such that the arc set of the resultant oriented graph is the disjoint-union of the arc sets of $(H^\tau \square G^\sigma)^o$ and $(H^\tau \otimes G^\sigma)^o$. Denote by $(H^\tau * G^\sigma)^o$ this resultant oriented graph and by \hat{S} be the skew-adjacency matrix of $(H^\tau * G^\sigma)^o$. The skew-spectrum of $(H^\tau * G^\sigma)^o$ is determined in the following theorem.

Theorem 2.6. *Let H^τ be an oriented bipartite graph of order m and let the skew-eigenvalues of H^τ be the non-zero values $\pm\mu_1i, \pm\mu_2i, \dots, \pm\mu_ti$ and $m - 2t$ 0's. Let G^σ be an oriented graph of order n and let the skew-eigenvalues of G^σ be the non-zero values $\pm\lambda_1i, \pm\lambda_2i, \dots, \pm\lambda_ri$ and $n - 2r$ 0's. Then the skew-eigenvalues of the oriented graph $(H^\tau * G^\sigma)^o$ are $\pm i\sqrt{(u_j^2 + 1)(\lambda_k^2 + 1) - 1}$ with multiplicities 2, $j = 1, \dots, t, k = 1, \dots, r, \pm\mu_ji$ with multiplicities $n - 2r, j = 1, \dots, t, \pm\lambda_ki$ with multiplicities $m - 2t, k = 1, \dots, r,$ and 0 with multiplicities $(m - 2t)(n - 2r)$.*

Proof. Suppose that (X, Y) is the bipartition of the vertices of H with $X = m_1$ and $Y = m_2$. Let S_1 and S_2 be the skew-adjacency matrices of H^τ and G^σ , respectively, where $S_1 = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$ and A is an $m_1 \times m_2$ matrix. Then the skew-adjacency matrix \hat{S} of $(H^\tau * G^\sigma)^o$ can be written as $\hat{S} = \bar{S} + S$, where \bar{S} and S are the skew-adjacency matrices of $(H^\tau \square G^\sigma)^o$ and $(H^\tau \otimes G^\sigma)^o$, respectively.

With suitable labeling of the vertices of $H * G$, we can derive the following formulas.

$$(2.1) \quad \bar{S} = I'_{m_1+m_2} \otimes S_2 + S_1 \otimes I_n \quad \text{and} \quad S = S'_1 \otimes S_2,$$

where $I'_{m_1+m_2} = \begin{pmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{pmatrix}$ and $S'_1 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$. For the details of Equation (2.1), one can also see Theorem 3.1 of [2] and Theorem 2.1 of this paper.

We then compute the singular values of \hat{S} . Note that $\hat{S}\hat{S}^T = \bar{S}\bar{S}^T + SS^T + \bar{S}S^T + S\bar{S}^T$. From Theorem 3.1 of [2] or direct computation, we can derive that $\bar{S}\bar{S}^T = -(I_m \otimes S_2^2 + S_1^2 \otimes I_n)$. It is obvious that $SS^T = (S'_1 \otimes S_2)((S'_1)^T \otimes S_2^T) = -(S'_1)^2 \otimes S_2^2 = S_1^2 \otimes S_2^2$. Moreover,

$$\begin{aligned} \bar{S}S^T + S\bar{S}^T &= (I'_{m_1+m_2} \otimes S_2 + S_1 \otimes I_n) (S'_1 \otimes (-S_2)) \\ &\quad + (S'_1 \otimes S_2) (I'_{m_1+m_2} \otimes (-S_2) + (-S_1) \otimes I_n) \\ &= - [(S_1 \otimes S_2^2 + S_1S'_1 \otimes S_2) + ((-S_1) \otimes S_2^2 + S'_1S_1 \otimes S_2)] \\ &= 0. \end{aligned}$$

To sum up all computation, we obtain that

$$\hat{S}\hat{S}^T = -I_m \otimes S_2^2 - S_1^2 \otimes I_n + S_1^2 \otimes S_2^2 = (S_1^2 - I_m) \otimes (S_2^2 - I_n) - I_{mn}.$$

Therefore, the eigenvalues of $\hat{S}\hat{S}^T$ are $(\mu^2 + 1)(\lambda^2 + 1) - 1$, where $\mu i \in Sp(H^\tau)$ and $\lambda i \in Sp(G^\sigma)$. Then the skew-spectrum of $(H^\tau * G^\sigma)^o$ immediately follows. The proof is complete. \square

Similar to Theorem 2.3, we can construct a new family of oriented graphs with the optimum skew energy by applying the above theorem.

Theorem 2.7. *Let H^τ be an oriented k -regular bipartite graph of order m with optimum skew energy $m\sqrt{k}$. Let G^σ be an oriented ℓ -regular graph of order n and optimum skew energy $n\sqrt{\ell}$. Then $(H^\tau * G^\sigma)^\circ$ is an oriented $(k + \ell + k\ell)$ -regular graph and has the optimum skew energy $\mathcal{E}_S((H^\tau * G^\sigma)^\circ) = mn\sqrt{k + \ell + k\ell}$.*

Comparing Theorems 2.3, 2.7 obtained above with Theorem 1.3 (or see Theorem 3.2 in [2]), we find that the oriented graphs constructed from these theorems have the same order mn but different regularities, which are $k\ell$, $k + \ell + k\ell$ and $k + \ell$, respectively.

Example 2.8. *Let $H = C_4$, $G_0 = K_4$, $G_1 = H \square G_0$, \dots , $G_r = H \square G_{r-1}$. Obviously, G_r is a $(2r + 3)$ -regular graph of order 4^{r+1} . From [1, 9], we know that H has the orientation with the optimum skew energy $4\sqrt{2}$ and G_0 has the orientation with the optimum skew energy $4\sqrt{3}$, see Figure 1. By Theorem 1.3, G_r has the orientation with the optimum skew energy $4^{r+1}\sqrt{2r + 3}$.*

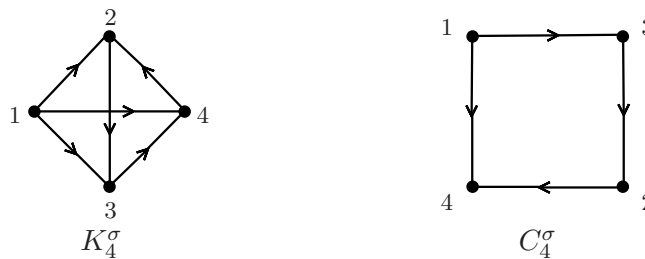


FIGURE 1. The orientations of K_4 and C_4 with the optimum skew energies

Example 2.9. *Let $H = C_4$, $G_0 = K_4$, $G_1 = H \otimes G_0$, \dots , $G_r = H \otimes G_{r-1}$. It is obvious that G_r is a $(3 \cdot 2^r)$ -regular graph of order 4^{r+1} . Then by Theorem 2.3, G_r has the orientation with optimum skew energy $4^{r+1}\sqrt{3 \cdot 2^r}$.*

Example 2.10. *Let $H = C_4$, $G_0 = K_4$, $G_1 = H \otimes G_0$, $G_2 = G_1 \otimes G_0$, \dots , $G_r = G_{r-1} \otimes G_0$. Note that $H, G_1, G_2, \dots, G_{r-1}$ are all regular bipartite graphs and G_r is a $(2 \cdot 3^r)$ -regular bipartite graph of order 4^{r+1} . Then by Theorem 2.3, G_r has the orientation with optimum skew energy $4^{r+1}\sqrt{2 \cdot 3^r}$.*

Example 2.11. *Let $H = C_4$, $G_0 = K_4$, $G_1 = H * G_0$, \dots , $G_r = H * G_{r-1}$. Note that G_r is a $(4 \cdot 3^r - 1)$ -regular graph of order 4^{r+1} . Then by Theorem 2.7, G_r has the orientation with optimum skew energy $4^{r+1}\sqrt{4 \cdot 3^r - 1}$.*

From Examples 2.9, 2.10 and 2.11, we can see that for some positive integers k , there exist oriented k -regular graphs with the optimum skew energy, which has order $n \leq k^2$. It is unknown that whether for any positive integer k , the oriented graph exists such that its order n is less than k^2 and it has an orientation with the optimum skew energy.

2.3. The orientation of $H[G]$. In this subsection, we consider the lexicographic product $H[G]$ of a bipartite graph H and a graph G . All definitions and notations are the same as above. We can see that the edge set $H[G]$ is the disjoint-union of the edge sets of $H \square G$ and $H \otimes K_n$, where K_n is a complete graph of order n .

Let H^τ and G^σ be oriented graphs of H and G with the skew-adjacency matrices S_1 and S_2 , respectively. Let K_n^ζ be an oriented graph of K_n with the skew-adjacency matrix S_3 . Then we can obtain two oriented graphs $(H^\tau \square G^\sigma)^\circ$ and $(H^\tau \otimes K_n^\zeta)^\circ$. Thus it is natural to yield an orientation of $H[G]$, denoted by $H[G]^\circ$, such that the arc set of $H[G]^\circ$ is the disjoint-union of the arc sets of $(H^\tau \square G^\sigma)^\circ$ and $(H^\tau \otimes K_n^\zeta)^\circ$. Let \tilde{S} be the skew-adjacency matrix of $H[G]^\circ$. We can see that $\tilde{S} = \bar{S} + S'_1 \otimes S_3$, where $\bar{S} = I'_{m_1+m_2} \otimes S_2 + S_1 \otimes I_n$ is the skew-adjacency matrix of $(H^\tau \square G^\sigma)^\circ$. Then

$$\begin{aligned} \tilde{S}^T \tilde{S} &= -\tilde{S}^2 = -(\bar{S} + S'_1 \otimes S_3)^2 \\ &= -[I_m \otimes S_2^2 + S_1^2 \otimes I_n + (S'_1)^2 \otimes S_3^2 + \bar{S}(S'_1 \otimes S_3) + (S'_1 \otimes S_3)\bar{S}] \\ &= -[I_m \otimes S_2^2 + S_1^2 \otimes I_n - S_1^2 \otimes S_3^2 + S_1 \otimes (S_2 S_3) - S_1 \otimes (S_3 S_2)] \end{aligned}$$

Suppose that H^τ is an oriented k -regular bipartite graph of order m with optimum skew energy $m\sqrt{k}$. Then $S_1^T S_1 = kI_m$. Let G^σ be an oriented ℓ -regular graph of order n and optimum skew energy $n\sqrt{\ell}$. Then $S_2^T S_2 = \ell I_n$. It is obvious that $H[G]^\circ$ is $(kn + \ell)$ -regular. Moreover, let K_n^ζ be an oriented graph of K_n with optimum skew energy $n\sqrt{n-1}$. Then $S_3^T S_3 = (n-1)I_n$, that is, S_3 is a skew-symmetric Hardamard matrix [8] of order n . If another condition that $S_2 S_3 = S_3 S_2$ holds, then

$$\tilde{S}^T \tilde{S} = -[I_m \otimes S_2^2 + S_1^2 \otimes I_n - S_1^2 \otimes S_3^2] = (kn + \ell)I_{mn}.$$

By Lemma 2.2, $H[G]^\circ$ has the optimum skew energy $mn\sqrt{kn + \ell}$.

The following example illustrates that the oriented graph satisfying the above conditions indeed exists.

Example 2.12. Let H^τ is an arbitrary oriented k -regular bipartite graph of order m with the optimum skew energy $m\sqrt{k}$. Let C_4^σ be the oriented graph of C_4 with optimum skew energy $4\sqrt{2}$ and the skew-adjacency matrix S_2 , and K_4^ζ be the oriented graph of K_4 with optimum skew energy $4\sqrt{3}$ and the skew-adjacency matrix S_3 , see Figure 1. It can be verified that $S_2 S_3 = S_3 S_2$. It follows that $(H[G])^\circ$ is an oriented $(4k + 2)$ -regular graph of order $4m$ with optimum skew energy $4m\sqrt{4k + 2}$.

There are many options for H , such as P_2 , C_4 , $K_{4,4}$, the hypercube Q_d and so on, which forms a new family of oriented graphs with the optimum skew energy.

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Xueliang Li

Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin 300071, P. R. China

Email: lxl@nankai.edu.cn

Huishu Lian

College of Science, China University of Mining and Technology, Xuzhou 221116, P. R. China

Email: lhs6803@126.com