www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 4 No. 4 (2015), pp. 5-14.
© 2015 University of Isfahan

www.ui.ac.ir

ON THE HARMONIC INDEX OF GRAPH OPERATIONS

B. SHWETHA SHETTY, V. LOKESHA* AND P. S. RANJINI

Communicated by Ivan Gutman

Abstract. The harmonic index of a connected graph G, denoted by $H(G)$, is defined as $H(G) = \sum_{uv\in E(G)} \frac{2}{d_u+d_v}$ where d_v is the degree of a vertex v in G. In this paper, expressions for the Harary indices of the join, corona product, Cartesian product, composition and symmetric difference of graphs are derived.

1. Introduction and Preliminaries

Throughout this paper we consider only simple connected graphs, i.e. connected graphs without loops and multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the set of all vertices and edges, respectively. For a graph G, the degree of a vertex v is the number of edges incident to v and denoted by d_v.

The *Harmonic index* $H(G)$ is vertex-degree-based topological index. This index first appeared in [5], and was defined as[14], [16]

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_u + d_v}$$

The connectivity index introduced in 1975 by Milan Randic [12], who has shown this index to reflect molecular branching. *Randic index* was defined as follows;

$$\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$$

MSC(2010): Primary: 05C20; Secondary: 05C05.

Keywords: Harmonic index, Graph operations.

Received: 29 October 2014, Accepted: 1 December 2014.

*Corresponding author.

Ernesto Estrada et al [4], introduced atom-bond connectivity (ABC) index, which it has been applied up until now to study the stability of alkanes and the strain energy of cycloalkanes. This index is defined as follows:

$$ABC(G) = \sum_{e=uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}$$

For a vertex v in a connected nontrivial graph G, The eccentricity $ecc_G(v)$ of a vertex v is the greatest geodesic distance between v and any other vertex. The Diameter $d(G)$ of G is defined as $d(G) = max\{ecc_G(v)|v \in V(G)\}$.

The *Composition* (also called lexicographic product [7]) $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_i, v_j) is adjacent with (u_k, v_l) whenever u_i is adjacent with u_k, or $u_i = u_k$ and v_j is adjacent with v_l.

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $v_j v_l \in E(G_2)$, or $u_i u_k \in E(G_1)$ and $v_j = v_l$ [2].

For given graphs G_1 and G_2 we define their *Corona product* $G_1 o G_2$ as the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $v_i \in V(G_1)$. Obviously, $|V(G_1 o G_2)| = |V(G_1)|(1 + |V(G_2)|)$ and $|E(G_1 o G_2)| = |E(G_1)| + |V(G_1)|(|V(G_2)| + |E(G_2)|)$[6].

A *sum* $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph on the vertex set $V(G_1) \cup V(G_2)$ and the edge set
$E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs[11].

The *symmetric difference* $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2)|u_1 v_1 \in E(G_1)$ or $u_2 v_2 \in E(G_2)$ but not both$\}$. Obviously,$|E(G_1 \oplus G_2)| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 4|E(G_1)||E(G_2)|$
$d_{G_1 \oplus G_2}(u, v) = |V(G_2)| d_{G_1}(u) + |V(G_1)| d_{G_2}(v) - 2 d_{G_1}(u) d_{G_2}(v)$ [8].

In this paper, expressions for the Harmonic indices of the join, corona product, Cartesian product, composition and symmetric difference of graphs are derived[1],[3],[10],[13],[15],[9].

2. Harmonic index of graph operations

Theorem 2.1. *Let G_1 and G_2 be two connected graphs with order n_1, n_2 and size m_1, m_2 respectively. Then*

$$H(G_1[G_2]) \leq \frac{1}{(1+n_2)^2} \Big\{ \frac{n_1}{2} ABC(G_2) + n_1 R(G_2) + n_1 n_2 m_2 + \frac{(n_2)^3}{2} ABC(G_1) + (n_2)^3 R(G_1) + (n_2)^2 m_1 \Big\}.$$

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ be a set of vertex for G_1 and G_2 respectively. By the definition of the composition of two graph one can see that,

$$|E(G_1[G_2])| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|$$
$$d_{G_1[G_2]}(u,v) = |V(G_2)|d_{G_1}(u) + d_{G_2}(v)$$

$$\begin{aligned}
H(G_1[G_2]) &= \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2]),(u_i,v_j)\neq(u_k,v_l)} \frac{2}{d_{G_1[G_2]}(u_i,v_j) + d_{G_1[G_2]}(u_k,v_l)} \\
&= \sum_{(u_i,v_j),(u_i,v_l)\in E(G_1[G_2]),j\neq l} \frac{2}{d_{G_1[G_2]}(u_i,v_j) + d_{G_1[G_2]}(u_i,v_l)} \\
&+ \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2]),i\neq k} \frac{2}{d_{G_1[G_2]}(u_i,v_j) + d_{G_1[G_2]}(u_k,v_l)} \\
&= \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{2}{|V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_j) + |V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_l)} \\
&+ \sum_{u_i,u_k\in E(G_1)} \sum_{v_j\in V(G_2)} \sum_{v_l\in V(G_2)} \frac{2}{|V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_j) + |V(G_2)|d_{G_1}(u_k) + d_{G_2}(v_l)} \\
&= \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{2}{d_{G_2}(v_j) + d_{G_2}(v_l) + 2n_2 d_{G_1}(u_i)} \\
&+ \sum_{u_i,u_k\in E(G_1)} \sum_{v_j\in V(G_2)} \sum_{v_l\in V(G_2)} \frac{2}{n_2 d_{G_1}(u_i) + n_2 d_{G_1}(u_k) + d_{G_2}(v_j) + d_{G_2}(v_l)} \\
&= A_1 + A_2 \qquad (1)
\end{aligned}$$

where A_1, A_2 are the sums of the above terms, in order.

For any vertex u, v, w of a graph G and for any positive integer n_1, n_2, n_3

$$\frac{1}{\frac{1}{(n_1+n_2+n_3)}(n_1 d_u + n_2 d_v + n_3 d_w)} \leq \frac{1}{(n_1+n_2+n_3)}\left(\frac{n_1}{d_u} + \frac{n_2}{d_v} + \frac{n_3}{d_w}\right)$$

with equality if and only if $d_u = d_v = d_w$.

Therefore

$$\begin{aligned}
A_1 &= \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{2}{d_{G_2}(v_j) + d_{G_2}(v_l) + 2n_2 d_{G_1}(u_i)} \\
&\leq \frac{2}{2+2n_2} \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{1}{2+2n_2}\Big(\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} + \frac{2n_2}{d_{G_1}(u_i)}\Big) \\
&= \frac{1}{1+n_2}\Big\{\frac{1}{2+2n_2} \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \Big(\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} - \frac{2}{d_{G_2}(v_j)d_{G_2}(v_l)}\Big) \\
&+ \frac{1}{1+n_2} \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{1}{d_{G_2}(v_j)d_{G_2}(v_l)} \\
&+ \frac{n_2}{1+n_2} \sum_{u_i\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{1}{d_{G_1}(u_i)}\Big\}
\end{aligned}$$

$$\leq \frac{1}{1+n_2}\Big\{\frac{1}{2+2n_2}\sum_{u_i\in V(G_1)}\sum_{v_j,v_l\in E(G_2)}\sqrt{\frac{d_{G_2}(v_l)+d_{G_2}(v_j)-2}{d_{G_2}(v_l)d_{G_2}(v_j)}}$$
$$+\frac{1}{1+n_2}\sum_{u_i\in V(G_1)}\sum_{v_j,v_l\in E(G_2)}\frac{1}{\sqrt{d_{G_2}(v_j)d_{G_2}(v_l)}}$$
$$+\frac{n_2}{1+n_2}\sum_{u_i\in V(G_1)}\sum_{v_j,v_l\in E(G_2)}1\Big\}$$
$$= \frac{1}{1+n_2}\Big\{\frac{n_1}{2+2n_2}ABC(G_2)+\frac{n_1}{1+n_2}R(G_2)+\frac{n_1n_2m_2}{1+n_2}\Big\} \tag{2}$$

Similarly

$$A_2 = \sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\frac{2}{n_2d_{G_1}(u_i)+n_2d_{G_1}(u_k)+d_{G_2}(v_j)+d_{G_2}(v_l)}$$
$$\leq \frac{2}{2+2n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\frac{1}{2+2n_2}\Big(\frac{n_2}{d_{G_1}(u_i)}+\frac{n_2}{d_{G_1}(u_k)}+\frac{1}{d_{G_2}(v_j)}+\frac{1}{d_{G_2}(v_j)}\Big)$$
$$= \frac{1}{1+n_2}\Big\{\frac{n_2}{2+2n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\Big(\frac{1}{d_{G_1}(u_i)}+\frac{1}{d_{G_1}(u_k)}-\frac{2}{d_{G_1}(u_i)d_{G_1}(u_k)}\Big)$$
$$+\frac{n_2}{1+n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\frac{1}{d_{G_1}(u_i)d_{G_1}(u_k)}$$
$$+\frac{1}{2+2n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\Big(\frac{1}{d_{G_2}(v_j)}+\frac{1}{d_{G_2}(v_j)}\Big)\Big\}$$
$$\leq \frac{1}{1+n_2}\Big\{\frac{n_2}{2+2n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\sqrt{\frac{d_{G_1}(u_i)+d_{G_1}(u_k)-2}{d_{G_1}(u_i)d_{G_1}(u_k)}}$$
$$+\frac{n_2}{1+n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}\frac{1}{\sqrt{d_{G_1}(u_i)d_{G_1}(u_k)}}$$
$$+\frac{1}{2+2n_2}\sum_{u_i,u_k\in E(G_1)}\sum_{v_j\in V(G_2)}\sum_{v_l\in V(G_2)}(1+1)\Big\}$$
$$= \frac{1}{1+n_2}\Big\{\frac{(n_2)^3}{2+2n_2}ABC(G_1)+\frac{(n_2)^3}{1+n_2}R(G_1)+\frac{(n_2)^2m_1}{1+n_2}\Big\} \tag{3}$$

From equation (1),(2)and (3) we get

$$H(G_1[G_2]) \leq \frac{1}{(1+n_2)^2}\Big\{\frac{n_1}{2}ABC(G_2)+n_1R(G_2)+n_1n_2m_2$$
$$+\frac{(n_2)^3}{2}ABC(G_1)+(n_2)^3R(G_1)+(n_2)^2m_1\Big\}.$$

□

Theorem 2.2. *Let G_1 and G_2 be two connected graphs with order n_1, n_2 and size m_1, m_2 respectively. Then*

$$H(G_1\times G_2) \leq \frac{1}{8}\Big\{n_1ABC(G_2)+2n_1R(G_2)+2n_1m_2$$

$$+ n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1 \Big\}.$$

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ be a set of vertex for G_1 and G_2 respectively. By the Definition of the cartesian product of two graph one can see that,

$$|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|$$
$$d_{G_1 \times G_2}(u, v) = d_{G_1}(u) + d_{G_2}(v)$$

$$\begin{aligned}
H(G_1 \times G_2) &= \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \times G_2),(u_i,v_j) \neq (u_k,v_l)} \frac{2}{d_{G_1 \times G_2}(u_i, v_j) + d_{G_1 \times G_2}(u_k, v_l)} \\
&= \sum_{(u_i,v_j),(u_i,v_l) \in E(G_1 \times G_2), v_j v_l \in E(G_2)} \frac{2}{d_{G_1 \times G_2}(u_i, v_j) + d_{G_1 \times G_2}(u_i, v_l)} \\
&+ \sum_{(u_i,v_j),(u_k,v_j) \in E(G_1 \times G_2), u_i u_k \in E(G_1)} \frac{2}{d_{G_1 \times G_2}(u_i, v_j) + d_{G_1 \times G_2}(u_k, v_j)} \\
&= \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{2}{d_{G_2}(v_j) + d_{G_2}(v_l) + 2d_{G_1}(u_i)} \\
&+ \sum_{v_j \in V(G_2)} \sum_{u_i, u_k \in E(G_1)} \frac{2}{d_{G_1}(u_i) + d_{G_1}(u_k) + 2d_{G_2}(v_j)} \\
&= B_1 + B_2 \qquad (4)
\end{aligned}$$

where B_1, B_2 are the sums of the above terms, in order.

$$\begin{aligned}
B_1 &= \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{2}{d_{G_2}(v_j) + d_{G_2}(v_l) + 2d_{G_1}(u_i)} \\
&\leq \frac{1}{8} \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \Big(\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} + \frac{2}{d_{G_1}(u_i)} \Big) \\
&= \frac{1}{8} \Big\{ \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \Big(\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} - \frac{2}{d_{G_2}(v_j) d_{G_2}(v_l)} \Big) \\
&+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{1}{d_{G_2}(v_j) d_{G_2}(v_l)} \\
&+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{1}{d_{G_1}(u_i)} \Big\} \\
&\leq \frac{1}{8} \Big\{ \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \sqrt{\frac{d_{G_2}(v_l) + d_{G_2}(v_j) - 2}{d_{G_2}(v_l) d_{G_2}(v_j)}} \\
&+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{1}{\sqrt{d_{G_2}(v_j) d_{G_2}(v_l)}} \\
&+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} 1 \Big\}
\end{aligned}$$

$$= \frac{1}{8}\Big\{n_1 ABC(G_2) + 2n_1 R(G_2) + 2n_1 m_2\Big\} \tag{5}$$

Similarly we get,

$$B_1 \leq \frac{1}{8}\Big\{n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1\Big\} \tag{6}$$

From equation (4),(5)and (6) we get

$$\begin{aligned} H(G_1 \times G_2) \leq \frac{1}{8}\Big\{&n_1 ABC(G_2) + 2n_1 R(G_2) + 2n_1 m_2 \\ &+ n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1\Big\}. \end{aligned}$$

□

Theorem 2.3. *For $i \in \{1,2\}$, let G_i be a graph of minimum degree δ_i, maximum degree Δ_i,order n_i and size m_i. Then*

$$H(G_1 o G_2) \geq \frac{m_1}{\Delta_1 + n_2} + \frac{m_2 n_1}{\Delta_2 + 1} + \frac{2n_1 n_2}{\Delta_1 + \Delta_2 + n_2 + 1}$$
$$H(G_1 o G_2) \leq \frac{m_1}{\delta_1 + n_2} + \frac{m_2 n_1}{\delta_2 + 1} + \frac{2n_1 n_2}{\delta_1 + \delta_2 + n_2 + 1} \quad .$$

Proof. The edges of $G_1 o G_2$ are partitioned into three subsets E_1, E_2 and E_3 as follows

$$E_1 = \{e \in E(G_1 o G_2), e \in E(G_1)\}$$
$$E_2 = \{e \in E(G_1 o G_2), e \in E(G_{2i}), i = 1, 2, \ldots, |V(G_1)|\}$$
$$E_3 = \{e \in E(G_1 o G_2), e = uv, u \in V(G_{2i}), i = 1, 2, \ldots, |V(G_1)| \; and \; v \in V(G_1)\}$$

and if u is a vertex of $G_1 o G_2$, then

$$d_{G_1 o G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & if \;\; u \in V(G_1) \\ d_{G_2}(u) + 1 & if \;\; u \in V(G_2) \end{cases}$$

Let $G_1 = (V_i, E_i)$, $i \in \{1,2\}$ and let $G_1 o G_2 = (V, E)$
we have

$$\begin{aligned} H(G_1 o G_2) &= \sum_{uv \in E(G_1 o G_2)} \frac{2}{d_{G_1 o G_2}(u) + d_{G_1 o G_2}(v)} \\ &= Q_1 + Q_2 + Q_3 \end{aligned} \tag{7}$$

where

$$\begin{aligned} Q_1 &= \sum_{uv \in E_1} \frac{2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \\ &\geq \frac{2m_1}{\Delta_1 + n_2 + \Delta_1 + n_2} = \frac{m_1}{\Delta_1 + n_2} \end{aligned} \tag{8}$$

$$\begin{aligned} Q_2 &= n_1 \sum_{uv \in E_2} \frac{2}{d_{G_2}(u) + 1 + d_{G_2}(v) + 1} \\ &\geq \frac{2n_1 m_2}{\Delta_2 + 1 + \Delta_2 + 1} = \frac{n_1 m_2}{\Delta_2 + 1} \end{aligned} \tag{9}$$

$$Q_3 = \sum_{uv \in E_3, u \in V_1 and v \in V_2} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + 1}$$

$$\geq \frac{2n_1n_2}{\Delta_1 + \Delta_2 + n_2 + 1} \tag{10}$$

From equation (7),(8), (9)and (10) we get

$$H(G_1oG_2) \geq \frac{m_1}{\Delta_1 + n_2} + \frac{m_2n_1}{\Delta_2 + 1} + \frac{2n_1n_2}{\Delta_1 + \Delta_2 + n_2 + 1}$$

Similarly we deduce the lower bound

$$H(G_1oG_2) \leq \frac{m_1}{\delta_1 + n_2} + \frac{m_2n_1}{\delta_2 + 1} + \frac{2n_1n_2}{\delta_1 + \delta_2 + n_2 + 1} \quad .$$

□

Theorem 2.4. *Let G_1 and G_2 be two connected graphs with order n_1, n_2 and size m_1, m_2 respectively. Then*

$$H(G_1 + G_2) \geq \frac{2}{m_1 + 2n_2 + 1} R(G_1) + \frac{2}{m_2 + 2n_1 + 1} R(G_2) + \frac{n_1n_2}{n_1 + n_2 - 1} \quad .$$

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ be a set of vertex for G_1 and G_2 respectively. By the Definition of the join of two graph one can see that, if u is a vertex of $G_1 + G_2$, then

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & if \;\; u \in V(G_1) \\ d_{G_2}(u) + |V(G_1)| & if \;\; u \in V(G_2) \end{cases}$$

Therefore,

$$H(G_1 + G_2) = \sum_{uv \in E(G_1+G_2)} \frac{2}{d_{G_1+G_2}(u) + d_{G_1+G_2}(v)}$$

$$= \sum_{uv \in E(G_1)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} + \sum_{uv \in E(G_2)} \frac{2}{d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1}$$

$$+ \sum_{u \in V(G_1), v \in V(G_2)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1}$$

$$= A_1 + A_2 + A_3 \tag{11}$$

where

$$A_1 = \sum_{uv \in E(G_1)} \frac{2}{d_{G_1}(u) + d_{G_1}(v) + 2n_2}$$

Since for each edge uv of G, we have $d_G(u) + d_G(v) \leq |E(G)| + 1$ with equality iff every other edge of G is adjacent to the edge uv. And $1 \leq \sqrt{d_u d_v}$ the equality hold iff $d_u = d_v = 1$. Hence

$$A_1 \geq \sum_{uv \in E(G_1)} \frac{2}{|E(G_1)| + 1 + 2n_2} \times \frac{1}{\sqrt{d_u d_v}}$$

$$= \frac{2}{m_1 + 1 + 2n_2} R(G_1) \tag{12}$$

Similarly

$$A_2 \geq \frac{2}{m_2+1+2n_1}R(G_2) \tag{13}$$

Since for any graph G with n vertices $d_u \leq n-1$, therefore

$$\begin{aligned} A_3 &= \sum_{u\in V(G_1), v\in V(G_2)} \frac{2}{d_{G_1}(u)+n_2+d_{G_2}(v)+n_1} \\ &\geq \sum_{u\in V(G_1), v\in V(G_2)} \frac{2}{n_1-1+n_2+n_2-1+n_1} \\ &= \frac{n_1n_2}{n_1+n_2-1} \end{aligned} \tag{14}$$

From equation (11),(12), (13)and (14) we get

$$H(G_1+G_2) \geq \frac{2}{m_1+2n_2+1}R(G_1) + \frac{2}{m_2+2n_1+1}R(G_2) + \frac{n_1n_2}{n_1+n_2-1} \quad .$$

□

Theorem 2.5. *For $i \in \{1,2\}$, let G_i be a graph of diameter $d(G_i)$, order n_i and size m_i. Then*

$$H(G_1 \oplus G_2) \geq \frac{n_2^2m_1+n_1^2m_2-4m_1m_2}{n_2(n_1-d(G_1))+n_1(n_2-d(G_2))-2(n_1-d(G_1))(n_2-d(G_2))} \quad .$$

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ be a set of vertex for G_1 and G_2 respectively.

$$\begin{aligned} H(G_1 \oplus G_2) &= \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1\oplus G_2)} \frac{2}{d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l)} \\ &= \sum_{v_j\in V(G_2)} \sum_{v_l\in V(G_2)} \sum_{u_i,u_k\in E(G_1)} \frac{2}{d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l)} \\ &+ \sum_{u_i\in V(G_1)} \sum_{u_k\in V(G_1)} \sum_{v_j,v_l\in E(G_2)} \frac{2}{d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l)} \\ &- \sum_{u_i,u_k\in E(G_1)} \sum_{v_j,v_l\in E(G_1)} \frac{2}{d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l)} \end{aligned} \tag{15}$$

As per the definition of the Symmetric difference of a graph

$$\begin{aligned} d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l) &= n_2[d_{G_1}(u_i)+d_{G_1}(u_k)] + n_1[d_{G_2}(v_j)+d_{G_2}(v_l)] \\ &- 2[d_{G_1}(u_i)d_{G_2}(v_j)+d_{G_1}(u_k)d_{G_2}(v_l)] \end{aligned}$$

Since for each vertex u of a graph G with n vertices, $d_G(u) \leq n - ecc_G(u)$ and the diameter $d(G) \geq ecc_G(u)$.

Therefore,

$$\begin{aligned} d_{G_1\oplus G_2}(u_i,v_j)+d_{G_1\oplus G_2}(u_k,v_l) &\leq n_2[n_1-ecc_{G_1}(u_i)+n_1-ecc_{G_1}(u_k)] \\ &+ n_1[n_2-ecc_{G_2}(v_j)+n_2-ecc_{G_2}(v_l)] \\ &- 2[(n_1-ecc_{G_1}(u_i))(n_2-ecc_{G_2}(v_j)) \end{aligned}$$

$$
\begin{aligned}
&+ (n_1 - ecc_{G_1}(u_k))(n_2 - ecc_{G_2}(v_l))] \\
&\leq n_2[2n_1 - 2d(G_1)] + n_1[2n_2 - 2d(G_2)] \\
&- 2[(n_1 - d(G_1))(n_2 - d(G_2)) + (n_1 - d(G_1))(n_2 - d(G_2))] \\
&= 2\{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]\}
\end{aligned}
$$

(16)

From equation (15)and (16) we get

$$
\begin{aligned}
H(G_1 \oplus G_2) \geq & \frac{n_2^2 m_1}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \\
& + \frac{n_1^2 m_2}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \\
& - \frac{4m_1 m_2}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \\
& = \frac{n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \quad .
\end{aligned}
$$

□

Acknowledgments

The authors wish to record their sincere thanks to the anonymous referees for carefully reading the manuscript and making suggestions that improve the content and presentation of the paper.

References

[1] M. Azari, Sharp lower bounds on the Narumi–Katayama index of graph operations, *Appl. Math. Comput.*, **239** (2014) 409–421.

[2] K. C. Das, K. Xu, I. N. Cangul, A. S. Cevik and A. Graovac, On the Harary index of graph operations, *J. Inequal. Appl.*, **2013** (2013) 1–16.

[3] N. De, A. Pal, S. Nayeem and M. Abu, *Connective eccentric index of some graph operations*, http://arXiv:1406.0378, 2014.

[4] E. Estrada, L. Torres, L. Rodríguez and I. Gutman, An Atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, *Indian J. Chem.*, **37** (1998) 849–855.

[5] S. Fajtlowicz, On conjectures of Graffiti. II, *Congr. Numer*, **60** (1987) 189–197.

[6] G. H. Fath-Tabar, A. Hamzeh and S. Hossein-Zadeh, GA_2 index of some graph operations, *Filomat*, **24** (2010) 21–28.

[7] F. Harary, *Graph theory*, Reading, MA: Addison-Wesley, 1994.

[8] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.*, **56** (2008) 1402–1407.

[9] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.*, **157** (2009) 804–811.

[10] V. Lokesha, B. S. Shetty, P. S. Ranjini, I. N. Cangul and A. S. Cevik, New bounds for Randic and GA indices, *J. Inequal. Appl.*, **2013** (2013) 1–7.

[11] K. Pattabiraman and M. Vijayaragavan, Reciprocal degree distance of some graph operations, *Trans. Comb.*, **2** (2013) 13–24.

[12] M. Randic, Characterization of molecular branching, *J. Am. Chem. Soc.*, **97** (1975) 6609–6615.

[13] B. Sh. Shetty, V. Lokesha, P. S. Ranjini and K. C. Das, Computing some topological indices of Smart polymer, *Digest Journal of Nanomaterials and Biostructures*, **7** (2012) 1097–1102.

[14] X. Xu, Relationships between harmonic index and other topological indices, *Appl. Math. Sci.*, **6** (2012) 2013–2018.

[15] I. G. Yero and J. A. Rodríguez-Velázquez, *On the Randić index of corona product graph*, International Scholarly Research Notices, 2011.

[16] L. Zhong, The harmonic index for graphs, *Appl. Math. Lett.*, **25** (2012) 561–566.

B. Shwetha Shetty

Don Bosco Institute of Technology, Bangalore-78, India

Email:shwethu17@gmail.com

V. Lokesha

Dept. of Mathematics, V S K University, Bellary - 583105, India

Email:lokiv@yahoo.com

P. S. Ranjini

Department of Mathematics, Don Bosco Institute of Technology, Bangalore-78, India

Email:ranjini_p_s@yahoo.com