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BROADCAST DOMINATION IN TORI

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ABSTRACT. A *broadcast* on a graph G is a function $f : V(G) \to \{0, 1, \ldots, diam(G)\}$ such that for every vertex $v \in V(G)$, $f(v) \leq e(v)$, where $diam(G)$ is the diameter of G, and $e(v)$ is the eccentricity of v. In addition, if every vertex hears the broadcast, then the broadcast is a *dominating broadcast*. The *cost* of a broadcast f is the value $\sigma(f) = \sum_{v \in V(G)} f(v)$. In this paper we determine the minimum cost of a dominating broadcast (also known as the *broadcast domination number*) for a torus $C_m \square C_n$.

1. Introduction

A radio station wishes to broadcast from towers of varying capacity so that the broadcast is heard by all intended recipients. Larger capacity towers can broadcast further, but will incur a higher associated cost (or transmission power, say in watts). To minimize this cost, the radio station has to place appropriate towers at carefully selected locations. As in [1], we can model the problem by a graph whose vertices are the sections of the region and where an edge between two vertices indicates that the two sections are close to each other.

Let G be a connected simple graph with vertex set $V(G)$. The *order* of G is the number of vertices in G. The *distance* between two vertices $u, v \in V(G)$, which we denote as $d(u, v)$, is the length of a shortest $u - v$ path in G. The *eccentricity* of a vertex $v \in V(G)$ is $e(v) = max\{d(v, u) \mid u \in V(G)\}$. The *radius* and *diameter* of G are defined as $rad(G) = min\{e(v) \mid v \in V(G)\}$ and $diam(G) = max\{e(v) \mid v \in V(G)\}$ respectively. A vertex $v \in V(G)$ is a *central vertex* if $e(v) = rad(G)$.

A function $f : V(G) \to \{0, 1, \ldots, diam(G)\}$ is a *broadcast* if $f(v) \leq e(v)$ for every vertex $v \in V(G)$. We define $V_f^+ = \{v \mid f(v) > 0\}$, and say that every vertex in V_f^+ is a *broadcast vertex*. We define the

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broadcast neighborhood of a broadcast vertex v as $N_f[v] = \{u \mid d(u,v) \leq f(v)\}$, and say that each vertex $u \in N_f[v]$ hears the broadcast f from vertex v. For a set $S \subseteq V_f^+$, we write $N_f[S] = \bigcup_{v \in S} N_f[v]$. If $N_f[V_f^+] = V(G)$, then f is a *dominating broadcast*. The *cost* of a broadcast f is denoted as $\sigma(f) = \sum_{v \in V_f^+} f(v)$. A dominating broadcast f with minimum cost is called a γ_b-*broadcast* of G, and the *broadcast domination number* for graph G, denoted by $\gamma_b(G)$, is defined as $\gamma_b(G) = \sigma(f)$, where f is a γ_b-*broadcast* of G. If $\gamma_b(G) = rad(G)$, then we say that G is a *radial graph*.

A graph H is called a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In addition, if $V(H) = V(G)$, then H is called a *spanning subgraph* of G.

Observation 1.1. *Let G and H be two connected graphs. If H is a spanning subgraph of G, then $\gamma_b(G) \leq \gamma_b(H)$.*

The Cartesian product of two graphs G_1 and G_2, denoted by $G = G_1 \,\square\, G_2$, has $V(G) = V(G_1) \times V(G_2) = \{(x_1, x_2) \mid x_i \in V(G_i) \text{ for } i = 1, 2\}$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$.

The problem of determining the broadcast domination number for the Cartesian product of two graphs was first studied in [2]. In that paper, the authors found a closed formula for the special case where the two graphs are paths, i.e., a grid graph $P_m \,\square\, P_n$. We present their result as follows.

Theorem 1.2. [2] *For $m \geq n \geq 2$, $\gamma_b(P_m \,\square\, P_n) = rad(P_m \,\square\, P_n) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.*

It is natural to consider other grid graphs such as the torus $C_m \,\square\, C_n$. By using a different approach from Theorem 1.2, we shall prove in this paper the following main result.

Theorem 1.3. *For $m, n \geq 3$, $\gamma_b(C_m \,\square\, C_n) = \lceil \frac{m+n}{2} \rceil - 1$.*

Notice that the broadcast domination numbers for $P_m \,\square\, P_n$ and $C_m \,\square\, C_n$ are consistent with Observation 1.1. In fact, the numbers are the same except for the case where both m and n are even.

The remainder of this paper is organized as follows. In Section 2, we state a known result on broadcast domination and review some results from optimization theory. These results are then used in Section 3 to prove Theorem 1.3.

2. Background

An *efficient broadcast* is a dominating broadcast such that every vertex hears from one broadcast vertex only (possibly including itself). It is clear that every graph G has an efficient broadcast, since we can broadcast from a central vertex with cost equal to $rad(G)$. We also have the following stronger result.

Theorem 2.1. [2] *Every graph G has a γ_b-broadcast which is efficient.*

We now introduce some definitions from optimization theory. A set C is said to be a *convex set* if for any $x, y \in C$ and any α in the interval $[0, 1]$, we have $(1-\alpha)x + \alpha y \in C$. Denote $\mathbb{R}$ to be the set of all real numbers. A real valued function $f : X \to \mathbb{R}$ defined on a convex set X is said to be *strictly convex* if for any two points x_1 and x_2 in X and any $\alpha \in [0,1]$, we have $f\big(\alpha x_1 + (1-\alpha)x_2\big) < \alpha f(x_1) + (1-\alpha) f(x_2)$. The following lemma is a direct consequence of the definition of a strictly convex function.

Lemma 2.2. *Let f be a strictly convex function on $\mathbb{R}$. Then we have $f(p) - f(p-1) < f(p+1) - f(p)$ for any integer p.*

Proof. Result follows after setting $x_1 = p-1$, $x_2 = p+1$ and $\alpha = \frac{1}{2}$ in the earlier definition of a strictly convex function. □

Lemma 2.3. *Let w and t be two positive integers such that $w \geq t$. Then $(w-t+1)^2 + t - 1$ is the objective value for the following optimization problem:*

$$\begin{aligned} \max \;\; & y_1^2 + y_2^2 + \cdots + y_t^2 \\ s.t. \;\; & y_1 + y_2 + \cdots + y_t = w, \\ & y_1, y_2, \ldots, y_t \textit{ are positive integers.} \end{aligned}$$

Proof. For some fixed integer t, let $\vec{y} = (y_1, y_2, \ldots, y_t)$. We first show that objective function $f(\vec{y}) = ||\vec{y}||^2$ is strictly convex. By the triangle inequality and the AM-GM inequality,

$$\begin{aligned} f\big(\alpha\vec{y}_1 + (1-\alpha)\vec{y}_2\big) &= ||\alpha\vec{y}_1 + (1-\alpha)\vec{y}_2||^2 \\ &\leq \Big(||\alpha\vec{y}_1|| + ||(1-\alpha)\vec{y}_2||\Big)^2 \\ &= \alpha^2||\vec{y}_1||^2 + 2\alpha(1-\alpha)||\vec{y}_1|| \cdot ||\vec{y}_2|| + (1-\alpha)^2||\vec{y}_2||^2 \\ &< \alpha^2||\vec{y}_1||^2 + \alpha(1-\alpha)\Big(||\vec{y}_1||^2 + ||\vec{y}_2||^2\Big) + (1-\alpha)^2||\vec{y}_2||^2 \\ &= \alpha f(\vec{y}_1) + (1-\alpha) f(\vec{y}_2). \end{aligned}$$

We next prove this lemma by induction on $w \geq t$. Without loss of generality, let $y_1 \geq y_2 \geq \cdots \geq y_t$. It is easy to see that result holds when $w = t$, and that when $w = t+1$, the optimal solution must be $\vec{y} = (2,1,1,\ldots,1)$. Thus the objective value is $2^2 + t - 1$. For $w = t+2$, the only feasible solutions are $\vec{y} = (3,1,1,\ldots,1)$, or $(2,2,1,1,\ldots,1)$. Hence by Lemma 2.2, the objective value is $3^2 + t - 1$. The proof is completed after repeating the above procedure for a finite number of times. □

3. The tori $C_m \,\square\, C_n$

We shall now prove Theorem 1.3 in this section. Note that $C_m \,\square\, C_n$ is isomorphic to $C_n \,\square\, C_m$.

Lemma 3.1. *Let $u \in V_f^+$ be any broadcast vertex of $C_n \,\square\, C_m$, where $m, n \geq 3$, and let $p = f(u)$. Then the maximum number of vertices that hear broadcast f from vertex u is $2p^2 + 2p + 1$.*

Proof. The number of vertices that hear the broadcast f is maximized when $min\{m,n\} \geq 2p+1$. This maximum value is equal to $1 + 3 + \cdots + (2p-1) + (2p+1) + (2p-1) + \cdots + 3 + 1 = 2p^2 + 2p + 1$. □

We shall first establish the broadcast domination numbers for the graph $C_n \,\square\, C_m$ in the two special cases where $0 \leq |m-n| \leq 1$.

Lemma 3.2. *Consider any broadcast vertex $u \in V_f^+$ for the graph $C_k \,\square\, C_k$, where $k \geq 3$. Let α be the number of vertices that hear broadcast f from vertex u, and let $p = f(u)$. Then*

$$\alpha = \begin{cases} 2p^2 + 2p + 1 & \text{if } k \geq 2p+1 \\ 4kp + 2k - 2p^2 - 2p - k^2 & \text{if } k \leq 2p \text{ and } k \text{ is odd} \\ 4kp + 2k - 2p^2 - 2p - k^2 - 1 & \text{if } k \leq 2p \text{ and } k \text{ is even.} \end{cases}$$

Proof. The value of α for the case where $k \geq 2p+1$ is proved in Lemma 3.1. It remains to show the values of α for the remaining two cases. First, we define the function $N(k)$ as follows:

$$N(k) = \underbrace{(2p+1) + (2p-1) + (2p-1) + (2p-3) + (2p-3) + \dots}_{k \text{ terms}}. \tag{3.1}$$

Observe that

$$\begin{aligned} N(k) &= (2p+2-1) + (2p+2-2) + \cdots + (2p+2-k) - \left\lfloor \frac{k}{2} \right\rfloor \\ &= 2k(p+1) - \frac{k(k+1)}{2} - \left\lfloor \frac{k}{2} \right\rfloor. \end{aligned}$$

Next, we compute the value of α when $k \leq 2p$. It is clear that α can be obtained by replacing each term that has value greater than k in (3.1) by the value k. Consider the first $(2p+2-k)$ terms in (3.1). When k is odd, summing the terms gives

$$\begin{aligned} &(2p+2-1) + \Big((2p+2-2) - 1\Big) + (2p+2-3) + \dots \\ &\quad + \Big(2p+2-(2p+2-k-1) - 1\Big) + \Big(2p+2-(2p+2-k)\Big) \\ &= (2p+2-1) + (2p+2-2) + \cdots + k + k. \end{aligned}$$

Notice that each of the first $(2p+2-k)$ terms in (3.1) is greater than or equal to k, and each of the remaining terms has value less than k. Thus it suffices to replace the first $(2p+2-k)$ terms in (3.1) by k to compute α as follows:

$$\begin{aligned} \alpha &= \underbrace{\big(k + k + \cdots + k\big)}_{2p+2-k \text{ terms}} + \underbrace{\big(2p+2-(2p+2-k+1)\big) + \dots}_{\text{remaining terms from (3.1)}} \\ &= k(2p+2-k) \ + \ N(k) \ - \ N(2p+2-k) \\ &= k(2p+2-k) + \left(2k(p+1) - \frac{k(k+1)}{2} - \left\lfloor \frac{k}{2} \right\rfloor\right) \\ &\quad - \left(2(2p+2-k)(p+1) - \frac{(2p+2-k)(2p+3-k)}{2} - \left\lfloor \frac{2p+2-k}{2} \right\rfloor\right) \\ &= \dots \\ &= 4kp + 2k - 2p^2 - 2p - k^2. \end{aligned}$$

The case when k is even is identical to the case above, and is omitted. □

Figure 1 illustrates the result of Lemma 3.2 using the graph $C_6 \,\Box\, C_6$.

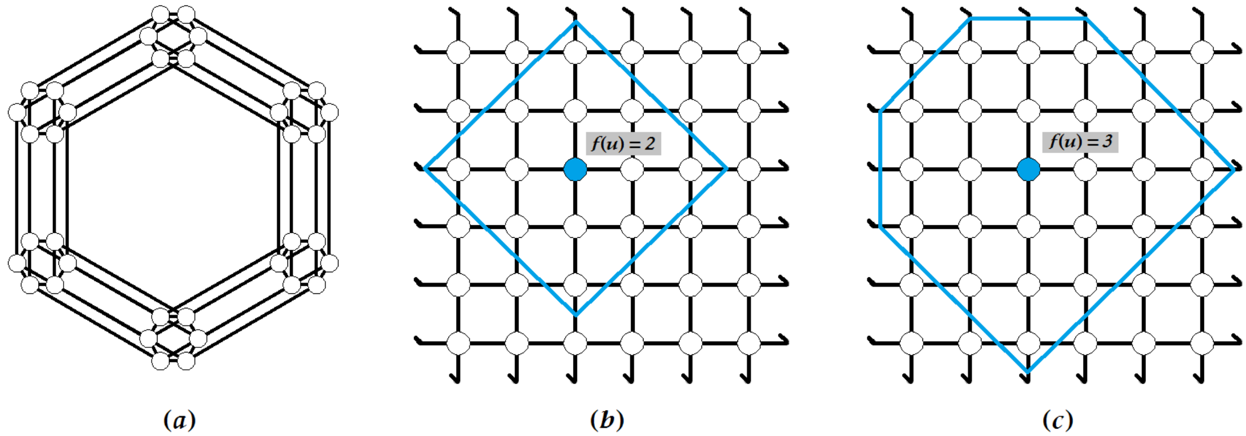

FIGURE 1. $C_6 \,\Box\, C_6$, with $\alpha = 13$, and 23 in (b) and (c) respectively.

Lemma 3.3. *For $k \geq 3$, $\gamma_b(C_k \,\Box\, C_k) = k - 1$.*

Proof. Notice that $rad(C_k \,\Box\, C_k) = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = k - 1$ when k is odd. Thus any vertex of $C_k \,\Box\, C_k$ can form a dominating broadcast. When k is even, we first pick any vertex (say u) of $C_k \,\Box\, C_k$. It can be verified easily that the following is a dominating broadcast:

$$g(v) = \begin{cases} k-2 & \text{if } v = u \\ 1 & \text{if } v \text{ satisfies } d(u,v) = rad(C_k \,\Box\, C_k) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\gamma_b(C_k \,\Box\, C_k) \leq k - 1$. To prove the equality, we assume on the contrary that $\gamma_b(C_k \,\Box\, C_k) \leq k - 2$. From Theorem 2.1, the graph $C_k \,\Box\, C_k$ has an efficient γ_b-broadcast, say f. Let $V_f^+ = \{u_1, u_2, \ldots, u_t\}$, where t is the number of broadcast vertices. Also, let s be the cardinality of the set $\{u_i \in V_f^+ \mid f(u_i) \geq \frac{k}{2}\}$. Clearly $s \in \{0, 1\}$, so we consider two cases.

Case 1: $s = 0$. Clearly $t \geq 2$. Then by Lemma 3.2, the maximum number of vertices that hear the broadcast f is the objective value of the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{t} \Big(2f(u_i)^2 + 2f(u_i) + 1\Big) \\ s.t. \quad & \sum_{i=1}^{t} f(u_i) = k - 2, \\ & 2 \leq t \leq k - 2, \\ & 1 \leq f(u_i) \leq \left\lfloor \frac{k-1}{2} \right\rfloor \quad for\ 1 \leq i \leq t. \end{aligned}$$

Since the objective function is strictly convex for any fixed t, the optimal solution can be obtained by induction on $\sum_{i=1}^{t} f(u_i)$. The approach is similar to that of Lemma 2.3. Start from $\sum_{i=1}^{t} f(u_i) = 2$

with $t = 2$, so that $\big(f(u_1), f(u_2)\big) = (1,1)$. Now consider the inductive step where $t = a$ for some integer $a \le k-2$. As $\sum_{i=1}^{t} f(u_i)$ increases by 1, we can either increase $f(u_i)$ by 1 for some positive integer $i \le a$, or increase the value of t by 1 and set $f(u_{a+1}) = 1$ if $a \neq k-2$.

Finally we obtain the optimal solution: $t = 2$, $\big(f(u_1), f(u_2)\big) = \big(\lceil \frac{k-2}{2} \rceil, \lfloor \frac{k-2}{2} \rfloor\big)$. By considering the case when k is even, we have $\sum_{i=1}^{2}\big(2f(u_i)^2 + 2f(u_i) + 1\big) = 2 \cdot \gamma_b(C_k \,\Box\, C_k) + 2 + \sum_{i=1}^{2}\big(2f(u_i)^2\big) \le 2(k-2)+2+2\big(\frac{k-2}{2}\big)^2 + 2\big(\frac{k-2}{2}\big)^2 = k^2 + (2-2k) < k^2$. Similarly when k is odd, we have $\sum_{i=1}^{2}\big(2f(u_i)^2 + 2f(u_i)+1\big) = 2 \cdot \gamma_b(C_k \,\Box\, C_k) + 2 + \sum_{i=1}^{2}\big(2f(u_i)^2\big) \le 2(k-2)+2+2\big(\frac{k-1}{2}\big)^2 + 2\big(\frac{k-3}{2}\big)^2 = k^2 + (3-2k) < k^2$. Since k^2 is the order of the graph $C_k \,\Box\, C_k$, f is not a dominating broadcast for all $k \ge 3$. This is a contradiction.

Case 2: $s = 1$. Without loss of generality, let u_t be the vertex satisfying $f(u_t) = p$, where $p \ge \frac{k}{2}$. By Lemma 3.2, the number of vertices that hear broadcast f from vertex u_t is at most $4kp + 2k - 2p^2 - 2p - k^2$. This implies that the number of vertices that do not hear broadcast f from vertex u_t is at least

$$k^2 - \big(4kp + 2k - 2p^2 - 2p - k^2\big) \ge 2(k-p)^2 - 2(k-p). \tag{3.2}$$

We now show that not all vertices hear the broadcast f when $t \ge 2$. From our earlier assumption that $\gamma_b(C_k \,\Box\, C_k) = p + \sum_{i=1}^{t-1} f(u_i) \le k-2$, we have $2 \le t \le 1 + \sum_{i=1}^{t-1} f(u_i) \le 1 + (k-2-p) = k-p-1$. Since $f(u_i) < \frac{k}{2}$ for $1 \le i \le t-1$, by Lemma 2.3, the maximum value of $\sum_{i=1}^{t-1} f(u_i)^2$ occurs when

$$\big(f(u_1), f(u_2), \ldots, f(u_{t-1}), f(u_t)\big) = (\underbrace{1,1,1,\ldots,1}_{t-2 \text{ terms}}, K, p),$$

where $K = \gamma_b(C_k \,\Box\, C_k) - p - (t-2)$. Thus we have from Lemma 3.2 that

$$\begin{aligned}
\sum_{i=1}^{t-1}\Big(1 + 2f(u_i) + 2f(u_i)^2\Big) &\le (t-1) + 2\big(\gamma_b(C_k \,\Box\, C_k) - p\big) + 2\Big(\big(\gamma_b(C_k \,\Box\, C_k) - p - (t-2)\big)^2 + (t-2)\Big) \\
&= (3t-5) + 2(k-p-2) + 2(k-p-t)^2 \\
&\le 3(k-p-1) - 5 + 2(k-p-2) + 2(k-p-2)^2 \qquad (3.3) \\
&= 2(k-p)^2 - 3(k-p) - 4 \\
&= 2(k-p)^2 - 2(k-p) + \big(-4 - (k-p)\big) \\
&< 2(k-p)^2 - 2(k-p). \qquad (3.4)
\end{aligned}$$

The inequalities in (3.3) and (3.4) hold because $t < k-p$ and $2 \le k-p-1$ respectively. It now follows from (3.2) and (3.4) that not all vertices hear the broadcast f. Thus $s = t = 1$, and so $C_k \,\Box\, C_k$ is radial. However, this contradicts our assumption that $\gamma_b(C_k \,\Box\, C_k) \le k-2$. ☐

Lemma 3.4. *For $k \ge 4$, $\gamma_b(C_{k-1} \,\Box\, C_k) = k-1$.*

Proof. We prove this lemma by induction on k. When $k = 4$, it is clear that $\gamma_b(C_3 \,\Box\, C_4) = 3$. Assume that the result is true for $k = r-1$, that is $\gamma_b(C_{r-2} \,\Box\, C_{r-1}) = r-2$. We want to show that the result holds for $k = r$, i.e. $\gamma_b(C_{r-1} \,\Box\, C_r) = r-1$. Since $\gamma_b(C_{r-1} \,\Box\, C_r) \le \gamma_b(C_r \,\Box\, C_r) = r-1$, it suffices to show that $\gamma_b(C_{r-1} \,\Box\, C_r) \ge r-1$.

Suppose on the contrary that we can find a dominating broadcast f with cost equal to $r-2$ for the graph $C_{r-1} \square C_r$. Let $V_f^+ = \{u_1, u_2, \dots\}$. We claim that there exists a broadcast vertex (say u_1) such that $f(u_1) \geq \frac{r}{2}$; for otherwise, $f(u_i) \leq \lfloor \frac{r-1}{2} \rfloor = \lceil \frac{r-2}{2} \rceil$ for all $u_i \in V_f^+$, and by considering the case when r is even, we have

$$\begin{aligned}\sum_{i=1}^{2}\Big(2f(u_i)^2 + 2f(u_i) + 1\Big) &= 2 \cdot \gamma_b(C_{r-1} \square C_r) + 2 + \sum_{i=1}^{2}\Big(2f(u_i)^2\Big) \\ &\leq 2(r-2) + 2 + 2\Big(\frac{r-2}{2}\Big)^2 + 2\Big(\frac{r-2}{2}\Big)^2 \\ &= r^2 + (2-2r) \\ &< r^2 + (3-2r).\end{aligned}$$

When r is odd, a similar result can be obtained, so that in both cases, $\sum_{i=1}^{2}\big(2f(u_i)^2 + 2f(u_i) + 1\big) \leq (r^2 - r) + (3 - r)$. This inequality contradicts our assumption that f is a dominating broadcast, as the maximum number of vertices that hear f is strictly less than $r^2 - r$, the order of the graph $C_{r-1} \square C_r$.

Next, we define a broadcast g that is obtained from f by setting $g(u_1) = f(u_1) - 1$, and shifting all broadcast vertices $V_f^+ \backslash \{u_1\}$ where necessary (see Figure 2).

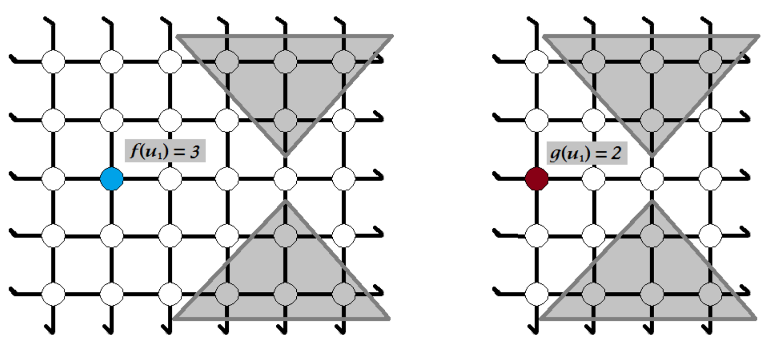

FIGURE 2. Dominating broadcast f of $C_5 \square C_6$ with $f(u_1) = 3$ (left), and dominating broadcast g of $C_5 \square C_4$ that is derived from f (right).

In Figure 2, the vertices in the shaded regions hear the respective broadcasts from $V_f^+ \backslash \{u_1\}$. It is clear that g is a dominating broadcast for the graph $C_{r-1} \square C_{r-2}$ with cost equal to $r-3$. This implies that $\gamma_b(C_{r-2} \square C_{r-1}) = r-3$, which is a contradiction. Hence the result holds for $k = r$. $\square$

We are now ready to prove Theorem 1.3, the main result in this paper.

Proof for Theorem 1.3. The proof is similar to that of Lemma 3.4. Assume without loss of generality that $m \geq n$. For any fixed $n \geq 3$, we shall prove the result by induction on m. By Lemmas 3.3 and 3.4, the statement is true when $m = n$, and when $m = n+1$. Now suppose that the statement is true for $n \leq m \leq r-1$, where $r \geq n+2$. We want to show that the statement is true for $m = s$, where s is either r or $r+1$. Notice that $s > n$. Suppose on the contrary that the statement is false, that is $\gamma_b(C_s \square C_n) \leq \lceil \frac{s+n}{2} \rceil - 2$. Then we can find a dominating broadcast f with cost equal to $\lceil \frac{s+n}{2} \rceil - 2$ for the graph $C_s \square C_n$. Let $V_f^+ = \{u_1, u_2, \dots, u_t\}$, where t is the number of broadcast vertices.

Claim: There exists a broadcast vertex (say u_1) such that $f(u_1) \geq \lceil \frac{n+1}{2} \rceil$.

If our claim is false, then $f(u_i) \leq \lfloor \frac{n}{2} \rfloor$ for all $u_i \in V_f^+$. By Lemma 3.2, the maximum number of vertices that hear the broadcast f is the objective value of the following optimization problem,

$$\begin{aligned} \max \quad & \sum_{i=1}^{t} \Big(2f(u_i)^2 + 2f(u_i) + 1 \Big) \\ s.t. \quad & \sum_{i=1}^{t} f(u_i) = \left\lceil \frac{s+n}{2} \right\rceil - 2, \\ & 2 \leq t \leq \left\lceil \frac{s+n}{2} \right\rceil - 2, \\ & 1 \leq f(u_i) \leq \left\lfloor \frac{n}{2} \right\rfloor \ \ for\ 1 \leq i \leq t. \end{aligned}$$

We will not show the method for solving this optimization problem, since it has already been explained in the proof of Lemma 3.3 (Case 1). The optimal solution is $t = \alpha$, where $\alpha = \left\lceil \frac{\lceil \frac{s+n}{2} \rceil - 2}{\lfloor \frac{n}{2} \rfloor} \right\rceil = \left\lceil \frac{\frac{s+n-4}{2}}{\lfloor \frac{n}{2} \rfloor} \right\rceil$, and

$$\Big(f(u_1), f(u_2), \ldots, f(u_\alpha) \Big) = \Big(\underbrace{\left\lfloor \frac{n}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{2} \right\rfloor}_{\alpha - 1 \text{ terms}}, \left\lceil \frac{s+n}{2} \right\rceil - 2 - (\alpha - 1) \left\lfloor \frac{n}{2} \right\rfloor \Big).$$

Since $f(u_\alpha) = \Big(\lceil \frac{s+n}{2} \rceil - 2 - (\alpha-1) \lfloor \frac{n}{2} \rfloor \Big) \leq \lfloor \frac{n}{2} \rfloor$, the objective value, which we denote as β, satisfies

$$\begin{aligned} \beta &= \sum_{i=1}^{t} \Big(2f(u_i)^2 + 2f(u_i) + 1 \Big) \\ &\leq min \Big\{ 2 \left\lfloor \frac{n}{2} \right\rfloor^2, 2 \Big(\left\lceil \frac{s+n}{2} \right\rceil - 2 - (\alpha - 1) \left\lfloor \frac{n}{2} \right\rfloor \Big)^2 \Big\} \\ &\qquad + 2(\alpha - 1) \left\lfloor \frac{n}{2} \right\rfloor^2 + 2 \Big(\left\lceil \frac{s+n}{2} \right\rceil - 2 \Big) + \alpha. \end{aligned} \tag{3.5}$$

We now show that if our claim is false, then we will arrive at a contradiction that β is strictly less than the number of vertices of graph $C_s \square C_n$. Recall that $s > n$. We shall consider cases in order to obtain an expression for α that does not involve any floor or ceiling function. For odd integers $n \geq 3$, let $s = a(n-1) + b$ such that $a \geq 1$ and $0 \leq b \leq n-2$. It follows that

$$\begin{aligned} \alpha &= \left\lceil \frac{s+n-4}{n-1} \right\rceil = \left\lceil \frac{(a+1)(n-1) + b - 3}{n-1} \right\rceil \\ &= \begin{cases} a & \text{if } n = 3,\ b \in \{0, 1\}, \text{ and } a \geq 2 \\ a+1 & \text{if } 0 \leq b \leq 3 \text{ for odd } n \geq 5, \text{ and } a \geq 1 \\ a+2 & \text{if } 4 \leq b \leq n-2 \text{ for odd } n \geq 5, \text{ and } a \geq 1. \end{cases} \end{aligned}$$

For even integers $n \geq 4$, we let $s = cn + d$ such that $c \geq 1$ and $0 \leq d \leq n-1$. Then

$$\alpha = \left\lceil \frac{s+n-4}{n} \right\rceil = \left\lceil \frac{(c+1)n+d-4}{n} \right\rceil$$
$$= \begin{cases} c & \text{if } n=4,\ d=0, \text{ and } c \geq 2 \\ c+1 & \text{if } d \in \{1,2,3\} \text{ when } n=4, \text{ or } 0 \leq d \leq 4 \text{ for even } n \geq 6 \\ c+2 & \text{if } 5 \leq d \leq n-1 \text{ for even } n \geq 6, \text{ and } c \geq 1. \end{cases}$$

Based on α, we have 6 cases to consider in total.

Case 1: $n = 3$, $b \in \{0, 1\}$, and $a \geq 2$.

Since $s = a(n-1) + b = 2a + b$ and $\alpha = a$, we can simplify (3.5) to obtain

$$\begin{aligned} \beta &\leq \left(2\left(\frac{n-1}{2}\right)^2 + 2(\alpha-1)\left(\frac{n-1}{2}\right)^2 + s + n - 3 + \alpha\right) - sn + sn \\ &= \frac{a}{2}(n-1)^2 + a + s(1-n) + (n-3) + sn \\ &= 3a - 2(2a+b) + sn \\ &< sn. \end{aligned}$$

Case 2: $b \in \{0, 1, 2, 3\}$ for odd $n \geq 5$, and $a \geq 1$.

Since $\alpha = a + 1$, we have from (3.5) that

$$\begin{aligned} \beta &\leq 2\left(\frac{s+n-3}{2} - (\alpha-1)\left(\frac{n-1}{2}\right)\right)^2 + 2(\alpha-1)\left(\frac{n-1}{2}\right)^2 + s + n - 3 + \alpha \\ &= \frac{1}{2}(n+b-3)^2 + \frac{a}{2}(n-1)^2 + (a+1)n + b - 2 \\ &= g_1(a) + sn, \end{aligned}$$

where $g_1(a) = \frac{a}{2}(n-1)^2 + (a+1)n - an(n-1) + \frac{1}{2}(n+b-3)^2 - bn + b - 2$. Using the fact that $b \in \{0, 1, 2, 3\}$, we have $\frac{1}{2}(n+b-3)^2 - bn + b - 2 = \frac{1}{2}(n^2 - 6n) + \frac{1}{2}(b-3)^2 + b - 2 \leq \frac{1}{2}(n^2 - 6n) + \frac{5}{2}$. Therefore,

$$\begin{aligned} g_1(a) &\leq \frac{a}{2}(n-1)^2 + (a+1)n - an(n-1) + \frac{1}{2}(n^2 - 6n) + \frac{5}{2} \\ &= \frac{1-a}{2}(n-1)^2 + a - n + 2 \\ &\leq 5 - 7a \\ &< 0. \end{aligned}$$

It follows that $\beta < sn$.

Case 3: $4 \leq b \leq n-2$ for odd $n \geq 5$, and $a \geq 1$.

Noting that $\alpha = a+2$, (3.5) simplifies to

$$\begin{aligned}\beta &\le 2\Big(\frac{s+n-3}{2} - (\alpha-1)\Big(\frac{n-1}{2}\Big)\Big)^2 + 2(\alpha-1)\Big(\frac{n-1}{2}\Big)^2 + s+n-3+\alpha \\ &= \frac{1}{2}(b-2)^2 + \frac{a+1}{2}(n-1)^2 + (a+1)(n-1) + a + b \\ &= g_2(a) + sn,\end{aligned}$$

where $g_2(a) = \frac{a+1}{2}(n-1)^2 + a + (a+1)(n-1) - an(n-1) + \frac{1}{2}(b-2)^2 + b - bn$. Since $b-n+2 \le 0$, we have $\frac{1}{2}(b-2)^2 + b - bn = \frac{b}{2}(b-n+2) + 2 - \frac{b}{2}(n+4) \le -2n-6$. It follows that $\beta < sn$, since

$$\begin{aligned}g_2(a) &\le \frac{a+1}{2}(n-1)^2 + a + (a+1)(n-1) - an(n-1) - 2n - 6 \\ &= \frac{1-a}{2}\Big((n-1)^2 - 2\Big) - n - 6 \\ &< 0.\end{aligned}$$

Case 4: $n = 4$, $d = 0$, and $c \ge 2$.

Since $s = cn + d = 4c$ and $\alpha = c$, it follows from (3.5) that

$$\begin{aligned}\beta &\le \Big(2\Big(\frac{n}{2}\Big)^2 + 2(\alpha-1)\Big(\frac{n}{2}\Big)^2 + s + n - 3 + \alpha\Big) - sn + sn \\ &= \frac{c}{2}n^2 + c + s(1-n) + (n-3) + sn \\ &= 1 - 3c + sn \\ &< sn.\end{aligned}$$

Case 5: $d \in \{1,2,3\}$ when $n = 4$, or $0 \le d \le 4$ for even $n \ge 6$.

We make use of the conditions that $n \ge 4$, $c \ge 1$, and $0 \le d \le 4$. Define the function $g_3(c) = \frac{c}{2}n^2 + (c+1)(n+1) - 3 - cn^2 + \frac{1}{2}(n+d-3)^2 - dn + d$. It was shown in the proof for Case 2 that $\frac{1}{2}(n+d-3)^2 - dn + d \le \frac{1}{2}(n^2 - 6n) + \frac{5}{2}$ when $0 \le d \le d$. Thus,

$$\begin{aligned}g_3(c) &\le \frac{c}{2}n^2 + (c+1)(n+1) - 3 - cn^2 + \frac{1}{2}(n^2 - 6n) + \frac{5}{2} \\ &= \Big(\frac{1-c}{2}\Big)\Big((n-1)^2 - 3\Big) + \frac{3}{2} - n \\ &< 0.\end{aligned}$$

Since $\alpha = c+1$, (3.5) can be written as

$$\begin{aligned}\beta &\le 2\Big(\frac{s+n-3}{2} - (\alpha-1)\Big(\frac{n}{2}\Big)\Big)^2 + 2(\alpha-1)\Big(\frac{n}{2}\Big)^2 + s + n - 3 + \alpha \\ &= \frac{1}{2}(n+d-3)^2 + \frac{1}{2}cn^2 + (c+1)(n+1) + d - 3 \\ &= g_3(c) + sn \\ &< sn.\end{aligned}$$

Case 6: $5 \le d \le n-1$ for even $n \ge 6$, and $c \ge 1$.

We define $g_4(c) = \frac{1}{2}(c+1)n^2 + (c+2) + (c+1)n - 3 - cn^2 + \frac{1}{2}(d-3)^2 + d - dn$. Since $d - n + 1 \leq 0$, we have $\frac{1}{2}(d-3)^2 + d - dn = \frac{1}{2}d(d-n+1) + \frac{9}{2} - \frac{d}{2}(n+5) \leq -\frac{5}{2}n - 8$. Hence we have

$$\begin{aligned} g_4(c) &\leq \frac{1}{2}(c+1)n^2 + (c+2) + (c+1)n - 3 - cn^2 - \frac{5}{2}n - 8 \\ &= \Big(\frac{1-c}{2}\Big)\Big((n-1)^2 - 3\Big) - \frac{1}{2}(n+16) \\ &< 0. \end{aligned}$$

Noting that $\alpha = c + 2$, (3.5) becomes

$$\begin{aligned} \beta &\leq 2\bigg(\frac{s+n-3}{2} - (\alpha-1)\Big(\frac{n}{2}\Big)\bigg)^2 + 2(\alpha-1)\Big(\frac{n}{2}\Big)^2 + \alpha + s + n - 3 \\ &= \frac{1}{2}(d-3)^2 + \frac{1}{2}(c+1)n^2 + (c+2) + (c+1)n + d - 3 \\ &= g_4(c) + sn \\ &< sn. \end{aligned}$$

All the 6 cases imply that f is not a dominating broadcast if our claim is not true. Thus there exists a broadcast vertex (say u_1) such that $f(u_1) \geq \lceil \frac{n+1}{2} \rceil$. Then by defining a new broadcast g (see Figure 2), we have $\gamma_b(C_{s-2} \,\Box\, C_n) \leq \lceil \frac{s+n}{2} \rceil - 3$. This is a contradiction since we assume in our induction hypothesis that $\gamma_b(C_{s-2} \,\Box\, C_n) = \lceil \frac{(s-2)+n}{2} \rceil - 1 = \lceil \frac{s+n}{2} \rceil - 2$. Hence $\gamma_b(C_s \,\Box\, C_n) \geq \lceil \frac{s+n}{2} \rceil - 1$.

Finally to show that $\gamma_b(C_s \,\Box\, C_n) = \lceil \frac{s+n}{2} \rceil - 1$, we let u be any vertex of $C_s \,\Box\, C_n$. If $rad(C_s \,\Box\, C_n) \neq \lceil \frac{s+n}{2} \rceil - 1$, then we consider the following broadcast g:

$$g(v) = \begin{cases} \lceil \frac{s+n}{2} \rceil - 2 & \text{if } v = u \\ 1 & \text{if } v \text{ satisfies } d(u,v) = rad(C_s \,\Box\, C_n) \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that g is a dominating broadcast. □

References

[1] D. Erwin, Dominating broadcasts in graph, *Bull. Inst. Comb. Appl.*, **42** (2004) 89–105.

[2] J. Dunbar, D. Erwin, T. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, Broadcasts in graphs, *Discrete Appl. Math.*, **154** (2006) 59–75.

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