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## THE RESISTANCE DISTANCE AND KIRCHHOFF INDEX OF THE $k$ -TH SEMI-TOTAL POINT GRAPHS

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ABSTRACT. The  $k$ -th semi-total point graph  $R^k(G)$  of a graph  $G$ , is a graph obtained from  $G$  by adding  $k$  vertices corresponding to each edge and connecting them to the endpoints of the edge considered. In this paper, we obtain formulas for the resistance distance and Kirchhoff index of  $R^k(G)$ .

### 1. Introduction

Klein and Randić [10] conceived the resistance distance between the vertices of a graph  $G$ , denoted by  $r_{uv}$ , defined to be equal to the effective electrical resistance between the vertices  $u$  and  $v$  of a graph  $G$ , with unit resistors taken over any edge of  $G$ . For trees  $r_{uv} = d_{uv}$ , the distance between two vertices  $u$  and  $v$ , and therefore the resistance distances are primarily of interest in the case of cycle containing graphs. Resistance distances and molecular structure descriptors based on them were much studied in the chemical literature [10] and also attracted the attention of mathematicians [3, 4, 18]. In analogy to the classical Wiener index, the Kirchhoff index of  $G$ , was defined by the sum of resistance distances of all pairs of vertices of a molecular graph,

$$Kf(G) = \sum_{\{u,v\} \subseteq V(G)} r_{uv}(G). \quad (1)$$

All graphs considered in this paper are simple and undirected. For a graph  $G$ , let  $A_G, B_G, D_G$  denote its adjacency matrix, incident matrix and diagonal matrix of vertex degrees, respectively. The matrices  $L_G = D_G - A_G$  and  $B_G B_G^T = D_G + A_G$  are called Laplacian matrix and signless Laplacian matrix of  $G$ , respectively.

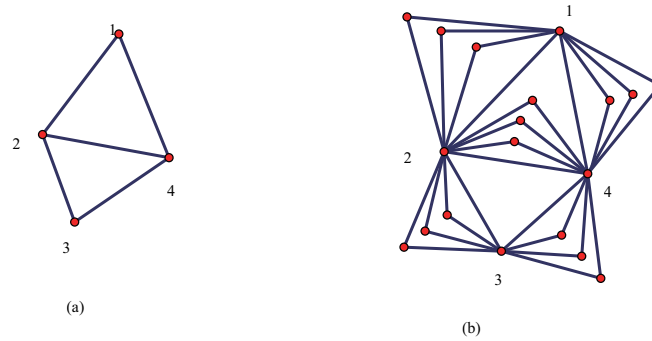
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FIGURE 1. The graph  $G$  and  $R^3(G)$ .

Resistance distances can be computed by the theory of resistance electrical network. The standard method to compute  $r_{uv}$  is via the (any) generalized inverse  $L_G^-$  of the Laplacian matrix  $L_G$  of a graph  $G$ . See Lemma 2.1 for details.

Let  $R(G)$  denote a graph obtained from  $G$  by adding a new vertex corresponding to each edge and connecting it to the endpoints of the edge considered. This graph is called the semi-total point graph. In [9], Joe et. al introduced a generalization of  $R(G)$ , denoted by  $R^k(G)$ : it is a graph obtained from  $G$  by adding  $k$  vertices corresponding to each edge and connecting them to the endpoints of the edge considered. This is equivalent to adding  $k$  triangles to each edge of  $G$ . We call  $R^k(G)$   $k$ -th semi-total point graph of  $G$ . For an edge  $uv$  of  $G$ , let  $(uv)^1, (uv)^2, \dots, (uv)^k$  be  $k$  new adding vertices corresponding to the edge  $uv$ . If  $G$  is a graph with  $n$  vertices and  $m$  edges, then  $R^k(G)$  has  $n + km$  vertices and  $m + 2km$  edges. In Figure 1, the graph  $G$  and its 3-th semi-toatl point graph  $R^3(G)$  are depicted.

It is of interest to study the Kirchhoff index of graphs derived from a single graph. The formulas and lower bounds of the Kirchhoff index of the line graph, subdivision graph,  $k$ -th semi-total point graph, and total graph of a connected regular graph were reported in [7, 14, 11]. The formulas for the resistance distance and Kirchhoff index of the subdivision graphs of general graphs were obtained in [15, 16]. The main aim of this paper is to give the resistance distance and Kirchhoff index of the  $k$ -th semi-total point graph of a general graph. This paper is organized as follows. In Section 2, some auxiliary lemmas are given. In Section 3, we obtain formulas for the resistance distance and Kirchhoff index of the  $k$ -th semi-total point graph of a general graph.

## 2. Preliminaries

The  $\{1\}$ -inverse of a matrix  $M$  is a matrix  $X$  such that  $MXM = M$ . If  $M$  is singular, then it has infinite  $\{1\}$ -inverses (see [10]). For a square matrix  $M$ , the group inverse of  $M$ , denoted by  $M^\#$ , is the unique matrix  $X$  such that  $MXM = M$ ,  $XM = X$  and  $MX = X$ . It is known that  $M^\#$  exists if and only if  $\text{rank}(M) = \text{rank}(M^2)$  (see [12, 13]). If  $M$  is real symmetric, then  $M^\#$  exists and  $M^\#$  is a symmetric  $\{1\}$ -inverse of  $M$  (see [10, 12, 14]).

In what follows we use  $M^{(1)}$  to denote any  $\{1\}$ -inverse of a matrix  $M$ . Let  $(M)_{uv}$  denote the  $(u, v)$ -entry of  $M$ .

**Lemma 2.1.** ([10]) *Let  $G$  be a connected graph. Then*

$$r_{uv} = (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} = (L_G^\#)_{uu} + (L_G^\#)_{vv} - 2(L_G^\#)_{uv}.$$

Let  $\mathbf{j}$  denote an all-one column vector.

**Lemma 2.2.** ([1]) *For any graph  $G$ , we have  $(L_G)^\#\mathbf{j} = 0$ .*

**Lemma 2.3.** (Foster's Formula [1]) *Let  $G$  be a connected graph with  $n$  vertices and edge set  $E(G)$ . Then  $\sum_{uv \in E(G)} r_{uv} = n - 1$ .*

Let  $Tr(M)$  denote the trace of a square matrix  $M$ .

**Lemma 2.4.** ([13]) *Let  $G$  be a connected graph on  $n$  vertices. Then*

$$Kf(G) = nTr(L_G^{(1)}) - \mathbf{j}^T L_G^{(1)} \mathbf{j} = nTr(L_G^\#).$$

**Lemma 2.5.** ([13]) *Let  $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$  be a symmetric matrix and  $A$  is nonsingular. Then  $N = \begin{pmatrix} A^{-1} + A^{-1}BS^\#B^T A^{-1} & -A^{-1}BS^\# \\ -S^\#B^T A^{-1} & S^\# \end{pmatrix}$  is a symmetric  $\{1\}$ -inverse of  $M$ , where  $S$  is the schur complement  $C - B^T A^{-1}B$  of  $A$  in  $M$ .*

### 3. The resistance distance and Kirchhoff index of $R^k(G)$

**Theorem 3.1.** *Let  $G$  be a connected graph with  $n$  vertices and  $m$  edges. Then the following hold:*

(1). *The Kirchhoff index of  $R^k(G)$  is*

$$\frac{(km + n)}{k + 2} \left( \frac{2}{n} Kf(G) + kTr(D_G L^\#) \right) + \frac{m^2 k^2 - n^2 + n}{2} + \frac{(mk + n)(n - 1)}{k + 2} - \frac{k^2}{2(k + 2)} \delta^T L_G^\# \delta,$$

where  $\delta = (d_1, d_2, \dots, d_n)^T$  is the degree sequence of  $G$ .

(2). *For any  $u, v \in V(G) \subseteq R^k(G)$ , we have  $r_{uv}(R^k(G)) = \frac{2}{k+2} r_{uv}(G)$ .*

(3). *For any  $e = (uv)^i, (uv \in E(G), i = 1, 2, \dots, k), w \in V(G)$ , we have*

$$r_{ew}(R^k(G)) = \frac{1}{2} + \frac{1}{k + 2} \left( r_{uw}(G) + r_{vw}(G) \right) - \frac{1}{2(k + 2)} r_{uv}(G).$$

(4). *For any  $e = (uv)^i \in E(G), f = (xy)^j, (uv, xy \in E(G), i, j = 1, 2, \dots, k), uv \neq xy$ , we have  $r_{ef}(R^k(G)) = 1 + \frac{1}{2(k+2)} \left( r_{ux}(G) + r_{uy}(G) + r_{vx}(G) + r_{vy}(G) - r_{uv}(G) - r_{xy}(G) \right)$ .*

(5). *For any  $e = (uv)^i, f = (uv)^j, (uv \in E(G), i, j = 1, 2, \dots, k)$ , we have  $r_{ef}(R^k(G)) = 1$ .*

*Proof.* With a suitable labeling for vertices of  $R^k(G)$ , the adjacency and degree matrices of  $R^k(G)$  can be written as follows:  $A = \begin{pmatrix} 0_{mk} & \Gamma^T \\ \Gamma & A_G \end{pmatrix}$ ,  $D = \begin{pmatrix} 2I_{mk} & 0 \\ 0 & (k+1)D_G \end{pmatrix}$ , where  $\Gamma = (B_G, B_G, \dots, B_G)^T$  is an  $(mk) \times n$  matrix and  $\Gamma\Gamma^T = k(D_G + A_G)$ . Hence, the Laplacian matrix of  $R^k(G)$  is  $L = \begin{pmatrix} 2I_{mk} & -\Gamma^T \\ -\Gamma & (k+1)D_G - A_G \end{pmatrix}$ .

Thus the Schur complement of  $2I_{mk}$  in  $L$  is

$$\begin{aligned} S &= (k+1)D_G - A_G - \Gamma(2I_{mk})^{-1}\Gamma^T \\ &= (k+1)D_G - A_G - \frac{k}{2}(D_G + A_G) \\ &= \frac{k+2}{2}(D_G - A_G) = \frac{k+2}{2}L_G. \end{aligned}$$

By Lemma 2.5, the following matrix

$$N = \frac{1}{2(k+2)} \begin{pmatrix} (k+2)I_{mk} + \Gamma^T L_G^\# \Gamma & 2\Gamma^T L_G^\# \\ 2L_G^\# \Gamma & 4L_G^\# \end{pmatrix} \tag{2}$$

is a symmetric  $\{1\}$ -inverse of  $L$ . By Lemma 2.1, we have

$$\begin{aligned} Kf(R^k(G)) &= (km+n)Tr(N) - \mathbf{j}^T N \mathbf{j} \\ &= (km+n) \left( \frac{2}{k+2} Tr(L_G^\#) + \frac{mk}{2} + \frac{1}{2(k+2)} Tr(\Gamma^T L_G^\# \Gamma) \right) - \mathbf{j}^T N \mathbf{j}. \end{aligned}$$

Since  $\Gamma^T L_G^\# \Gamma = \begin{pmatrix} B_G^T L_G^\# B_G^T & \cdots & B_G^T L_G^\# B_G^T \\ \vdots & \ddots & \vdots \\ B_G^T L_G^\# B_G^T & \cdots & B_G^T L_G^\# B_G^T \end{pmatrix}$  is a  $k \times k$  block matrix, we have

$$\begin{aligned} Tr(\Gamma^T L_G^\# \Gamma) &= kTr(B_G^T L_G^\# B_G) \\ &= \sum_{uv \in E(G)} \left( (L_G^\#)_{uu} + (L_G^\#)_{vv} + 2(L_G^\#)_{uv} \right) \\ &= \sum_{uv \in E(G)} \left( 2(L_G^\#)_{uu} + 2(L_G^\#)_{vv} - 2r_{uv}(G) \right) \\ &= 2Tr(D_G L_G^\#) - 2(n-1). \end{aligned}$$

Notice that  $B_G \mathbf{j} = (d_1, d_2, \dots, d_n)^T = \delta$  is the degree sequence of  $G$ , we have

$$\begin{aligned} \mathbf{j}^T N \mathbf{j} &= \frac{mk}{2} + \frac{1}{2(k+2)} \mathbf{j}^T \Gamma^T L_G^\# \Gamma \mathbf{j} \\ &= \frac{mk}{2} + \frac{k^2}{2(k+2)} \mathbf{j}^T B_G^T L_G^\# B_G \mathbf{j} \\ &= \frac{mk}{2} + \frac{k^2}{2(k+2)} \delta^T L_G^\# \delta. \end{aligned}$$

Therefore,

$$\begin{aligned} Kf(R^k(G)) &= (km+n)\left(\frac{2}{k+2}Tr(L_G^\#) + \frac{mk}{2} + \frac{1}{2(k+2)}Tr(\Gamma^T L_G^\# \Gamma)\right) - \mathbf{j}^T N \mathbf{j} \\ &= (km+n)\left(\frac{2}{k+2}Tr(L_G^\#) + \frac{k(km+n)}{k+2}Tr(D_G L_G^\#)\right) + \frac{m^2 k^2 - n^2 + n}{2} \\ &\quad + \frac{(mk+n)(n-1)}{k+2} - \frac{k^2}{2(k+2)}\delta^T L_G^\# \delta. \end{aligned}$$

For  $u, v \in V(G)$ , by Lemma 2.1 and equation (2), we have

$$r_{uv}(R^k(G)) = \frac{2}{k+2}r_{uv}(G).$$

For  $e = (uv)^i, (uv \in E(G), i = 1, 2, \dots, k), w \in V(G)$ , by Lemma 2.1 and equation (2), we have

$$\begin{aligned} r_{ew}(R^k(G)) &= \left(\frac{1}{2}I_{mk} + \frac{1}{2(k+2)}\Gamma^T L_G^\# \Gamma\right)_{ee} + \frac{1}{2(k+2)}(L_G^\#)_{ww} - \frac{2}{2(k+2)}(\Gamma^T L_G^\#)_{ew} \\ &= \frac{1}{2} + \frac{1}{2(k+2)}((L_G^\#)_{uu} + (L_G^\#)_{vv} + 2(L_G^\#)_{uv}) + \frac{1}{2(k+2)}(L_G^\#)_{ww} \\ &\quad - \frac{1}{2(k+2)}((L_G^\#)_{uw} + (L_G^\#)_{vw}) \\ &= \frac{1}{2} + \frac{1}{(k+2)}((r_{uw}(G) + r_{vw}(G)) - \frac{1}{2(k+2)}r_{uv}(G)). \end{aligned}$$

For  $e = (uv)^i, f = (xy)^j, (uv, xy \in E(G), i, j = 1, 2, \dots, k)$ , and  $uv \neq xy$ , by Lemma 2.1 and equation (2), we have

$$\begin{aligned} r_{ef}(R^k(G)) &= \frac{1}{2(k+2)}[((k+2)I_{mk} + \Gamma^T L_G^\# \Gamma)_{ee} + ((k+2)I_{mk} + \Gamma^T L_G^\# \Gamma)_{ff} - \Gamma^T L_G^\# \Gamma_{ef}] \\ &= 1 + \frac{1}{2(k+2)}(\Gamma^T L_G^\# \Gamma)_{ee} + \Gamma^T L_G^\# \Gamma_{ff} - \frac{1}{k+2}(\Gamma^T L_G^\# \Gamma)_{ef} \\ &= 1 + \frac{1}{2(k+2)}((B_G^T L_G^\# B_G)_{ee} + B_G^T L_G^\# B_G)_{ff} - \frac{1}{k+2}(B_G^T L_G^\# B_G)_{ef} \\ &= 1 + \frac{1}{2(k+2)}((L_G^\#)_{uu} + (L_G^\#)_{vv} + 2(L_G^\#)_{uv}) + (L_G^\#)_{vx} + \\ &\quad \frac{1}{2(k+2)}((L_G^\#)_{xx} + (L_G^\#)_{yy} + 2(L_G^\#)_{xy}) - \frac{1}{k+2}((L_G^\#)_{ux} + (L_G^\#)_{uy} + (L_G^\#)_{vy}) \\ &= 1 + \frac{1}{2(k+2)}(r_{ux}(G) + r_{uy}(G) + r_{vx}(G) + r_{vy}(G) - r_{uv}(G) - r_{xy}(G)). \end{aligned}$$

For  $e = (uv)^i, f = (uv)^j, (uv \in E(G), i, j = 1, 2, \dots, k)$ , and  $i \neq j$ , by Lemma 2.1 and equation (2), we have

$$\begin{aligned} r_{ef}(R^k(G)) &= \frac{1}{2(k+2)}[((k+2)I_{mk} + \Gamma^T L_G^\# \Gamma)_{ee} + ((k+2)I_{mk} + \Gamma^T L_G^\# \Gamma)_{ff} - \Gamma^T L_G^\# \Gamma_{ef}] \\ &= 1 + \frac{1}{2(k+2)}((B_G^T L_G^\# B_G)_{ee} + B_G^T L_G^\# B_G)_{ff} - \frac{1}{k+2}(B_G^T L_G^\# B_G)_{ef} \\ &= 1 + \frac{1}{2(k+2)}((B_G^T L_G^\# B_G)_{gg} + B_G^T L_G^\# B_G)_{gg} - \frac{1}{k+2}(B_G^T L_G^\# B_G)_{gg} \\ &= 1. \end{aligned}$$

This completes the proof. □

**Corollary 3.2.** ([11]) *Let  $G$  be a  $d$ -regular connected graph with  $n$  vertices. Then*

$$Kf(R^k(G)) = \frac{(dn + 2)^2}{2(k + 2)}Kf(G) + \frac{(n^2 - n)(kd + 2)}{2(k + 2)} + \frac{n^2(d^2k^2 - 4)}{8} + \frac{n}{2}.$$

*Proof.* For a  $d$ -regular graph  $G$  with  $n$  vertices, we have  $m = \frac{nd}{2}, D_G = dI_n, \delta = d\mathbf{j}_n$ . Thus  $D_GL^\# = dL^\#, \delta^T L_G^\# \delta = 0$  and the result holds. □

**Corollary 3.3.** ([14]) *Let  $G$  be a  $d$ -regular connected graph with  $n$  vertices. Then*

$$Kf(R(G)) = \frac{(dn + 2)^2}{6}Kf(G) + \frac{(n^2 - n)(d + 2)}{6} + \frac{n^2(d^2 - 4)}{8} + \frac{n}{2}.$$

The formula for the Kirchhoff index of  $R^k(G)$  in Theorem 3.1 is not as good as  $Tr(D_GL_G^\#)$  and  $\delta^T L_G^\# \delta$  are not easy to compute. Recently, two modifications of Kirchhoff index, which takes the degrees of the graph into account, have been considered. One is the *multiplicative degree-kirchhoff index* defined by Chen and Zhang [4]:

$$Kf^*(G) = \sum_{\{u,v\} \subseteq V} d_u d_v r_{uv}(G). \tag{3}$$

The other one is the *additive degree-Kirchhoff index* defined by Gutman et al. [6, 8].

$$Kf^+(G) = \sum_{\{u,v\} \subseteq V} (d_u + d_v)r_{uv}(G). \tag{4}$$

It is not difficult to check that  $Tr(D_GL_G^\#) = Kf^*(G)$ . It would be better if we express the  $Kf(R^k(G))$  in terms of  $Kf(G), Kf^*(G), Kf^+(G)$ . The following is such an expression.

**Theorem 3.4.** *Let  $G$  be a connected graph with  $n$  vertices and  $m$  edges. Then the Kirchhoff index of  $R^k(G)$  is*

$$\frac{1}{k + 2} \left( 2Kf(G) + kKf^+(G) + \frac{k^2}{2}Kf^*(G) \right) + \frac{k(k + 2)m^2 - k(k + 1)m + 2mn - n^2 + n}{2(k + 2)}.$$

*Proof.* Let the vertex set  $V(R^k(G)) = V(G) \cup \bigcup_{r=1}^k E^r$ , where  $E^r = \{(uv)^r : uv \in E(G)\}, r = 1, 2, \dots, k$ . By the definition of the Kirchhoff index, we have

$$\begin{aligned} Kf(R^k(G)) &= \sum_{\{i,j\} \subseteq V(R^k(G))} r_{ij}(R^k(G)) \\ &= \sum_{\{i,j\} \subseteq V(G)} r_{ij}(R^k(G)) + \sum_{\substack{i \in \bigcup_{r=1}^k E^r \\ j \in V(G)}} r_{ij}(R^k(G)) + \sum_{r=1}^k \sum_{\substack{i \in E^r \\ j \in E^r}} r_{ij}(R^k(G)) \\ &\quad + \sum_{\substack{i \in E^r \\ j \in E^s \\ r \neq s}} r_{ij}(R^k(G)) \end{aligned} \tag{5}$$

By Theorem 3.1, we can obtain the first and the last terms in summation (5):

$$\sum_{\{i,j\} \subseteq V(G)} r_{ij}(R^k(G)) = \sum_{\{i,j\} \subseteq V(G)} \frac{2}{k+2} r_{ij}(G) = \frac{2}{k+2} Kf(G). \tag{6}$$

$$\sum_{\substack{i \in E^r \\ j \in E^s \\ r \neq s}} r_{ij}(R^k(G)) = \sum_{\substack{i \in E^r \\ j \in E^s \\ r \neq s}} 1 = m \binom{k}{2}. \tag{7}$$

For the second term in summation (5), by the definition of  $R^k(G)$ , we have

$$\begin{aligned} \sum_{\substack{i \in \bigcup_{r=1}^k E^r \\ j \in V(G)}} r_{ij}(R^k(G)) &= k \sum_{\substack{i \in E^1 \\ j \in V(G)}} r_{ij}(R^k(G)) \\ &= k \sum_{\substack{e=uv \in E(G) \\ w=u \text{ or } v}} r_{ew}(R^k(G)) + k \sum_{\substack{e=uv \in E(G) \\ w \neq u,v}} r_{ew}(R^k(G)). \end{aligned}$$

By Foster’s formula, we have

$$k \sum_{\substack{e=uv \in E(G) \\ w=u \text{ or } v}} r_{ew}(R^k(G)) + \sum_{uv \in E(G)} r_{uv}(R^k(G)) = \sum_{\{i,j\} \subseteq V(R^k(G))} r_{ij}(R^k(G)) = km + n - 1.$$

Hence

$$k \sum_{\substack{e=uv \in E(G) \\ w=u \text{ or } v}} r_{ew}(R^k(G)) = km + n - 1 - \frac{2}{k+2}(n-1). \tag{8}$$

$$\begin{aligned} k \sum_{\substack{e=uv \in E(G) \\ w \neq u,v}} r_{ew}(R^k(G)) &= k \sum_{\substack{uv \in E(G) \\ w \neq u,v}} \left( \frac{1}{2} + \frac{r_{uw}(G) + r_{vw}(G) - \frac{1}{2}r_{uv}(G)}{k+2} \right) \\ &= km \frac{n-2}{2} + \frac{k}{k+2} \sum_{uv \in E(G)} \left( \sum_{w \neq u,v} r_{uw}(G) + \sum_{w \neq u,v} r_{vw}(G) - \frac{1}{2} \sum_{w \neq u,v} r_{uv}(G) \right) \\ &= km \frac{n-2}{2} + \frac{k}{k+2} \sum_{uv \in E(G)} \left( \sum_{w \in V(G)} r_{uw} + \sum_{w \in V(G)} r_{vw} - \frac{n+2}{2} r_{uv}(G) \right) \\ &= km \frac{n-2}{2} + \frac{k}{k+2} \sum_{uv \in E(G)} \sum_{u \in V(G)} d_u \sum_{w \in V(G)} r_{uw} + \frac{k}{k+2} \frac{n+2}{2} (n-1) \\ &= km \frac{n-2}{2} + \frac{k}{k+2} \sum_{\{u,v\} \subseteq V(G)} (d_u + d_v) r_{uv}(G) + \frac{k}{k+2} \frac{n+2}{2} (n-1) \\ &= km \frac{n-2}{2} + \frac{k}{k+2} Kf^+(G) + \frac{k}{k+2} \frac{n+2}{2} (n-1). \end{aligned} \tag{9}$$

Thus we obtain the second term in summation (5). In what follows we compute the third term in summation (5). By the definition of  $R^k(G)$ , we have

$$\begin{aligned}
 & \sum_{r=1}^k \sum_{\substack{i \in E^r \\ j \in E^r}} r_{ij}(R^k(G)) = k^2 \sum_{\substack{e=uv \in E(G) \\ f=xy \in E(G) \\ uv \neq xy}} r_{ef}(R^k(G)) \\
 &= k^2 \sum_{\substack{e=uv \in E(G) \\ f=xy \in E(G) \\ uv \neq xy}} \left( 1 + \frac{1}{2(k+2)} (r_{ux}(G) + r_{uy}(G) + r_{vx}(G) + r_{vy}(G) - r_{uv}(G) - r_{xy}(G)) \right) \\
 &= \frac{k^2 m(m-1)}{2} + \frac{k^2}{2(k+2)} \sum_{\{uv,xy\} \subseteq E(G)} (r_{ux}(G) + r_{uy}(G) + r_{vx}(G) + r_{vy}(G)) \\
 &\quad - \frac{k^2}{2(k+2)} \sum_{\{uv,xy\} \subseteq E(G)} (r_{uv}(G) + r_{xy}(G)). \tag{10}
 \end{aligned}$$

Let  $T = \sum_{\{uv,xy\} \subseteq E(G)} (r_{ux}(G) + r_{uy}(G) + r_{vx}(G) + r_{vy}(G))$ . For any two vertices  $u, v \in V$ , we compute the times that  $r_{uv}(G)$  appears in the summation of  $T$ . If  $uv \notin E(G)$ , then there are  $d_u d_v$  different pairs of edges in which one edge is incident to  $u$  and the other edge is incident to  $v$ . Thus  $r_{uv}(G)$  appears  $d_u d_v$  times in  $T$ . If  $uv \in E(G)$ , then there are  $d_u d_v - 1$  different pairs of edges as above. Thus  $r_{uv}(G)$  appears  $d_u d_v - 1$  times in  $T$ . Thus

$$\begin{aligned}
 T &= \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \notin E(G)}} d_u d_v r_{uv}(G) + \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \in E(G)}} (d_u d_v - 1) r_{uv}(G) \\
 &= \sum_{\{u,v\} \subseteq V(G)} d_u d_v r_{uv}(G) - \sum_{uv \in E(G)} r_{uv} \\
 &= K f^*(G) - (n-1). \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{\{uv,xy\} \subseteq E(G)} (r_{uv}(G) + r_{xy}(G)) = \frac{1}{2} \sum_{uv \in E(G)} \sum_{\substack{xy \in E(G) \\ xy \neq uv}} (r_{uv}(G) + r_{xy}(G)) \\
 &= \frac{1}{2} \sum_{uv \in E(G)} \left( (m-1)r_{uv} + \sum_{xy \in E(G)} r_{xy}(G) - r_{uv}(G) \right) \\
 &= \frac{1}{2} \sum_{uv \in E(G)} ((m-2)r_{uv} + n-1) \\
 &= \frac{1}{2} ((m-2)(n-1) + m(n-1)) = mn - m - n + 1. \tag{12}
 \end{aligned}$$

Thus we obtain the third term in summation (5). Substituting these terms back into summation (5) and some calculations yield Theorem 3.3. □

For  $k = 1$ , we have

**Corollary 3.5.** ([16]) *Let  $G$  be a connected graph with  $n$  vertices and  $m$  edges. Then the Kirchhoff index of  $R(G)$  is  $\frac{1}{3} \left( 2K f(G) + K f^+(G) + 2K f^*(G) \right) + \frac{3m^2 - 2m + 2mn - n^2 + n}{6}$ .*



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