



## WEIGHTED SZEGED INDICES OF SOME GRAPH OPERATIONS

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**ABSTRACT.** In this paper, the weighted Szeged indices of Cartesian product and Corona product of two connected graphs are obtained. Using the results obtained here, the weighted Szeged indices of the hypercube of dimension  $n$ , Hamming graph,  $C_4$  nanotubes, nanotorus, grid,  $t$ -fold bristled, sunlet, fan, wheel, bottleneck graphs and some classes of bridge graphs are computed.

### 1. Introduction

All the graphs considered in this paper are simple. A vertex  $x \in V(G)$  is said to be *equidistant* from the edge  $e = uv$  of  $G$  if  $d_G(u, x) = d_G(v, x)$ , where  $d_G(u, x)$  denotes the distance between  $u$  and  $x$  in  $G$ ; otherwise,  $x$  is a nonequidistant vertex. The degree of a vertex  $x \in V(G)$  is denoted by  $d_G(x)$ .

For an edge  $uv = e \in E(G)$ , the number of vertices of  $G$  whose distance to the vertex  $u$  is less than the distance to the vertex  $v$  in  $G$  is denoted by  $n_u(e, G)$ ; analogously,  $n_v(e, G)$  is the number of vertices of  $G$  whose distance to the vertex  $v$  in  $G$  is less than the distance to the vertex  $u$ ; the vertices equidistant from both the ends of the edge  $e = uv$  are not counted.

The two topological indices, namely, the Szeged index and weighted Szeged index of  $G$ , denoted by  $Sz(G)$  and  $Sz_w(G)$ , respectively, are defined as follows:

$$Sz(G) = \sum_{e=uv \in E(G)} n_u(e, G) n_v(e, G),$$

$$Sz_w(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v)) n_u(e, G) n_v(e, G).$$

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The *Cartesian product*  $G \square H$  of the graphs  $G$  and  $H$  has the vertex set  $V(G \square H) = V(G) \times V(H)$  and  $(u, x)(v, y)$  is an edge of  $G \square H$  if  $u = v$  and  $xy \in E(H)$  or,  $uv \in E(G)$  and  $x = y$ . Notation and definitions which are not given here can be found in [1] or [6]. For graphs  $G$  and  $H$  the *corona product*  $G \circ H$  of two graphs  $G$  and  $H$  is obtained by taking  $|V(G)|$  copies of  $H$  and joining each vertex of  $i$ th copy with vertex  $v_i \in V(G)$ .

The Szeged index studied by Gutman [4], Gutman and Dobrynin [5] and Khadikar et. al. [9] is closely related to the Wiener index of a graph. Basic properties of Szeged index and its analogy to the Wiener index are discussed by Klavžar et. al.[8]. It is proved that for a tree  $T$  the Wiener index of  $T$  is equal to its Szeged index. Ashrafi et. al. [10] have explained the differences between Szeged and Wiener indices of graphs. The mathematical properties and chemical applications of Szeged index are well studied by Dobrynin et. al. [2], Gutman et. al. [3] and Randic et. al. [19]. Recently Pisanski and Randic [18] studied the measuring network bipartivity using Szeged index. Weighted vertex PI index and weighted Szeged index of graph  $G$  has been introduced by Ilić and Milosavljevic [11], also they obtained the upper and lower bounds for weighted vertex PI index of graph. Also the exact formula for weighted vertex PI index of Cartesian product of graphs is obtained in [11]. Weighted vertex PI indices of generalized hierarchical product and corona product of two graphs are obtained in [17]. Some topological indices of the corona product of two graphs are studied in [12, 13, 14]. In this paper, the weighted Szeged indices of Cartesian product and corona product of two connected graphs are obtained. Using the results obtained here, the weighted Szeged indices of the hypercube of dimension  $n$ , Hamming graph,  $C_4$  nanotubes, nanotorus, grid,  $t$ -fold bristled, sunlet, fan, wheel, bottleneck graphs and some classes of bridge graphs are computed.

## 2. Weighted Szeged index of $G \square H$

In this section, we compute the weighted Szeged index of Cartesian product of two graphs. The following lemma is used in the proof of the main theorem of this section.

**Lemma 2.1.** *Let  $G$  and  $H$  be two connected graphs. Then*

- (i)  $|V(G \square H)| = |V(G)||V(H)|$ ,  $|E(G \square H)| = |E(G)||V(H)| + |E(H)||V(G)|$ .
- (ii)  $d_{G \square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')$ .
- (iii)  $d_{G \square H}((g, h)) = d_G(g) + d_H(h)$ .

For  $e = uv \in E(G)$ , let  $T_G(e; u) = \{x \in V(G) \mid d_G(u, x) < d_G(v, x)\}$  and let  $T_G(e; v) = \{x \in V(G) \mid d_G(v, x) < d_G(u, x)\}$ .

**Theorem 2.2.** *Let  $G$  and  $H$  be two connected graphs. Then  $Sz_w(G \square H) = |V(H)|^3 Sz_w(G) + |V(G)|^3 Sz_w(H) + 4|V(H)|^2 |E(H)| Sz(G) + 4|V(G)|^2 |E(G)| Sz(H)$ .*

**Proof.** Let  $V(G) = \{u_1, u_2, \dots, u_n\}$  and let  $V(H) = \{v_1, v_2, \dots, v_m\}$ . For our convenience, we partition the edge set of  $G \square H$  into two sets,  $E_1 = \{(u_r, v_i)(u_r, v_k) \mid u_r \in V(G), v_i v_k \in E(H)\}$  and  $E_2 = \{(u_r, v_i)(u_s, v_i) \mid u_r u_s \in E(G), v_i \in V(H)\}$ , that is,  $E_1 = \cup_{u_i \in V(G)} E(\langle X_i \rangle)$  and  $E_2 = \cup_{j=1}^m E(\langle Y_j \rangle)$ ,

that is,  $E_1$  denotes the edges of the copies of  $H$  corresponding to the vertices of  $G$  and,  $E_2$  denotes the edges of the copies of  $G$  corresponding to the vertices of  $H$ .

Let  $e = v_i v_k \in E(H)$  and let  $v_j \in T_H(e; v_i)$ . Then, for any  $u_r \in V(G)$  and  $e' \in E_1 \subset E(G \square H)$ , the distance of  $(u_r, v_i)$  to each vertex of  $Y_j$ , is less than its distance to the vertex  $(u_r, v_k)$  in  $G \square H$ . It can be observed that if some vertex  $v_s \notin T_H(e, v_i)$ , then all the vertices of the column  $Y_s$  are not in  $T_{G \square H}(e'; (u_r, v_i))$  in  $G \square H$ . Also if  $v_r$  is equidistant to  $e$  in  $H$ , then every vertex of  $Y_r$  is equidistant to  $e'$ . Consequently, for the edge  $e' \in E_1$  (of  $G \square H$ ) corresponding to  $e$  (in  $H$ ),

$$(2.1) \quad n_{(u_r, v_i)}(e', G \square H) = |V(G)| n_{v_i}(e, H)$$

$$(2.2) \quad \text{and similarly, } n_{(u_r, v_k)}(e', G \square H) = |V(G)| n_{v_k}(e, H).$$

Hence for  $E_1$  defined as above,

$$\begin{aligned} & \sum_{(u_r, v_i)(u_r, v_k) = e' \in E_1} \left( d_{G \square H}((u_r, v_i)) + d_{G \square H}((u_r, v_k)) \right) n_{(u_r, v_i)}(e', G \square H) n_{(u_r, v_k)}(e', G \square H) \\ = & \sum_{(u_r, v_i)(u_r, v_k) = e' \in E_1} \left( d_G(u_r) + d_H(v_i) + d_G(u_r) + d_H(v_k) \right) |V(G)| n_{v_i}(e, H) |V(G)| n_{v_k}(e, H), \\ & \text{by (2.1) and (2.2), where } e = v_i v_k \in E(H), \\ = & |V(G)| \sum_{v_i v_k = e \in E(H)} |V(G)|^2 (d_H(v_i) + d_H(v_k)) n_{v_i}(e, H) n_{v_k}(e, H) \\ & + 2 \sum_{u_r \in V(G)} d_G(u_r) \sum_{v_i v_k = e \in E(H)} |V(G)|^2 n_{v_i}(e, H) n_{v_k}(e, H), \text{ since } |E_1| = |V(G)| |E(H)|, \\ (2.3) \Rightarrow & |V(G)|^3 Sz_w(H) + 4 |V(G)|^2 |E(G)| Sz(H). \end{aligned}$$

Since the Cartesian product is commutative, similar to (2.1) and (2.2) we can see that

$$(2.4) \quad n_{(u_i, v_\ell)}(e', G \square H) = |V(H)| n_{u_i}(e, G)$$

$$(2.5) \quad n_{(u_k, v_\ell)}(e', G \square H) = |V(H)| n_{u_k}(e, G).$$

Hence

$$\begin{aligned} & \sum_{(u_i, v_\ell)(u_k, v_\ell) = e' \in E_2} \left( d_{G \square H}((u_i, v_\ell)) + d_{G \square H}((u_k, v_\ell)) \right) n_{(u_i, v_\ell)}(e', G \square H) n_{(u_k, v_\ell)}(e', G \square H) \\ = & \sum_{(u_i, v_\ell)(u_k, v_\ell) = e' \in E_2} \left( d_G(u_i) + d_H(v_\ell) + d_G(u_k) + d_H(v_\ell) \right) |V(H)| n_{u_i}(e, G) |V(H)| n_{u_k}(e, G), \\ & \text{by (2.4) and (2.5), where } e = u_i u_k \in E(G) \\ = & |V(H)| \sum_{u_i u_k = e \in E(G)} |V(H)|^2 (d_G(u_i) + d_G(u_k)) n_{u_i}(e, G) n_{u_k}(e, G) + \\ & + 2 \sum_{v_\ell \in V(H)} d_H(v_\ell) \sum_{u_i u_k = e \in E(G)} |V(H)|^2 n_{u_i}(e, G) n_{u_k}(e, G), \\ & \text{since } |E_2| = |V(H)| |E(G)|, \\ (2.6) \Rightarrow & |V(H)|^3 Sz_w(G) + 4 |V(H)|^2 |E(H)| Sz(G). \end{aligned}$$

Now we shall obtain the  $Sz_w(G \square H)$ . By definition,

$$\begin{aligned}
 Sz_w(G \square H) &= \sum_{(x,y)(a,b) = e' \in E(G \square H)} \left( d_{G \square H}((x, y)) + d_{G \square H}((a, b)) \right) (n_{(x,y)}(e', G \square H) n_{(a,b)}(e', G \square H)) \\
 &= \sum_{(x,y)(x,b) = e' \in E_1} \left( d_{G \square H}((x, y)) + d_{G \square H}((a, b)) \right) (n_{(x,y)}(e', G \square H) n_{(x,b)}(e', G \square H)) \\
 &\quad + \sum_{(x,y)(a,y) = e' \in E_2} \left( d_{G \square H}((x, y)) + d_{G \square H}((a, b)) \right) (n_{(x,y)}(e', G \square H) n_{(a,y)}(e', G \square H)) \\
 &= |V(H)|^3 Sz_w(G) + |V(G)|^3 Sz_w(H) + 4|V(H)|^2 |E(H)| Sz(G) \\
 &\quad + 4|V(G)|^2 |E(G)| Sz(H), \text{ by (2.3) and (2.6).}
 \end{aligned}$$

Let  $G_1, G_2, \dots, G_n$  be graphs with vertex set  $V(G_i)$  and edge set  $E(G_i)$ ,  $1 \leq i \leq n$ . Denote by  $\prod_{i=1}^n G_i$  the Cartesian product of graphs  $G_1, G_2, \dots, G_n$ . Clearly,  $|V(\prod_{i=1}^n G_i)| = \prod_{i=1}^n |V(G_i)|$ . By induction on  $n$ , one can see that  $|E(\prod_{i=1}^n G_i)| = \prod_{i=1}^n |V(G_i)| \sum_{i=1}^n \frac{|E(G_i)|}{|V(G_i)|}$ . In [8], S, Klavžar et al. have proved  $Sz(\prod_{i=1}^n G_i) = \sum_{i=1}^n Sz(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^3$ .

Next we compute a similar result for the weighted Szeged index.

**Theorem 2.3.** *Let  $G_1, G_2, \dots, G_n$  be connected graphs. Then  $Sz_w(\prod_{i=1}^n G_i) = \sum_{i=1}^n Sz_w(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^3 + 4 \sum_{i,j=1, i \neq j}^n Sz(G_i) |V(G_j)|^2 |E(G_j)| \prod_{k=1, i \neq k \neq j}^n |V(G_k)|^3$ .*

**Proof.** The proof follows by mathematical induction. For  $n = 2$ , the proof follows from Theorem 2.2. Assume that the result hold for  $n$  graphs. Then

$$\begin{aligned}
 Sz_w(\prod_{i=1}^{n+1} G_i) &= Sz_w(\prod_{i=1}^n G_i \square G_{n+1}) \\
 &= |V(\prod_{i=1}^n G_i)|^3 Sz_w(G_{n+1}) + |V(G_{n+1})|^3 Sz_w(\prod_{i=1}^n G_i) \\
 &\quad + 4 \left( |V(\prod_{i=1}^n G_i)|^2 |E(\prod_{i=1}^n G_i)| Sz(G_{n+1}) + |V(G_{n+1})|^2 |E(G_{n+1})| Sz(\prod_{i=1}^n G_i) \right) \\
 &= Sz_w(G_{n+1}) \prod_{i=1}^n |V(G_i)|^3 + |V(G_{n+1})|^3 \sum_{i=1}^n Sz_w(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^3 \\
 &\quad + 4 \sum_{i,j=1, i \neq j}^n Sz(G_i) |V(G_j)|^2 |E(G_j)| \prod_{k=1, i \neq k \neq j}^n |V(G_k)|^3 \\
 &\quad + 4 \left( Sz(G_{n+1}) \sum_{i=1}^n |V(G_i)|^2 |E(G_i)| \prod_{j=1, j \neq i}^n |V(G_j)|^3 + |V(G_{n+1})| |E(G_{n+1})| \sum_{i=1}^n Sz(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^3 \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^{n+1} Sz_w(G_i) \prod_{j=1, j \neq i}^{n+1} |V(G_j)|^3 + 4 \left( \sum_{i,j=1, i \neq j}^n Sz(G_i) |V(G_j)|^2 |E(G_j)| \prod_{k=1, i \neq k \neq j}^{n+1} |V(G_k)|^3 \right. \\
 &\quad \left. + \sum_{i \leq j \leq n} Sz(G_i) |V(G_j)|^2 |E(G_j)| \prod_{k=1, i \neq k \neq j}^{n+1} |V(G_k)|^3 \right) \\
 &= \sum_{i=1}^n Sz_w(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^3 + 4 \sum_{i,j=1, i \neq j}^n Sz(G_i) |V(G_j)|^2 |E(G_j)| \prod_{k=1, i \neq k \neq j}^n |V(G_k)|^3.
 \end{aligned}$$

The proof of the following corollary directly follows from Theorem 2.3.

**Corollary 2.4.** *Let  $G$  be a connected graph. Then  $Sz_w(G^n) = Sz_w(\prod_{i=1}^n G) = n |V(G)|^{3n-4} \{ |V(G)| Sz_w(G) + 4(n-1) |E(G)| Sz(G) \}$ .*

**Example 2.5.** *Suppose  $Q_n$  denotes the hypercube of dimension  $n$ . Then by Theorem 2.3,  $Sz_w(Q_n) = Sz_w(K_2^n) = n^2 2^{(3n-2)}$ .*

Let us consider the graph  $G$  whose vertices are the  $N$ -tuples  $b_1, b_2, \dots, b_N$  with  $b_i \in \{0, 1, \dots, n_i - 1\}, n_i \geq 2$ , and two vertices be adjacent if the corresponding tuples differ in precisely one place; such a graph is called a *Hamming graph*. It is well-known fact that a graph  $G$  is a Hamming graph if and only if it can be written in the form  $G = \prod_{i=1}^N K_{n_i}$  and so the Hamming graph is usually denoted by  $H_{n_1, n_2, \dots, n_N}$ . In the following lemma, the weighted Szeged index of a Hamming graph is computed.

It is easy to see that  $Sz(K_n) = \frac{n(n-1)}{2}$  and  $Sz_w(K_n) = n(n-1)^2$ . The proof of the following lemma follows from Theorem 2.3.

**Lemma 2.6.** *Let  $G$  be a Hamming graph with above parameter. Then  $Sz_w(H_{n_1 n_2 \dots n_N}) = \left( \sum_{i=1}^N \left( 1 - \frac{1}{n_i} \right)^2 + \sum_{i,j=1, i \neq j}^N \frac{(n_i-1)(n_j-1)}{n_i^2} \right) \prod_{i=1}^N n_i^3$ .*

Let  $C_n$  and  $P_n$  denote the cycle and path on  $n$  vertices, respectively. It is known that  $Sz(C_n) = \frac{n^3}{4}$  when  $n$  is even, and  $\frac{n(n-1)^2}{4}$  otherwise and  $Sz(P_n) = \binom{n+1}{3}$ ; see [7]. It can be easily verified that  $Sz_w(C_n) = n^3$  when  $n$  is even, and  $n(n-1)^2$  otherwise and  $Sz_w(P_n) = \frac{2(n-1)(n^2+n-3)}{3}$ .

Using Theorems 2.2, 2.3 and  $Sz_w(P_n), Sz_w(C_n), Sz(P_n)$  and  $Sz(C_n)$ , we obtain the exact weighted Szeged indices of the following graphs.

**Example 2.7.** *The graphs  $L_n = P_n \square K_2, R = P_n \square C_m, S = C_m \square C_n$  and  $T = P_m \square P_n$  are known as ladder,  $C_4$  nanotubes,  $C_4$  nanotorus and grid, respectively. The exact weighted Szeged indices of these graphs are given below.*

1.  $Sz_w(L_n) = 14n^3 - 4n^2 - 24n + 16$ .
2.  $Sz_w(R) = \begin{cases} \frac{m^3}{3}(10n^3 - 3n^2 - 10n + 6) & \text{if } m \text{ is even} \\ \frac{2m^3}{3}(n^3 + n^2 - 5n + 3) + m(m-1)^2 n^2 (2n-1) & \text{if } m \text{ is odd.} \end{cases}$

$$3. Sz_w(S) = \begin{cases} 2n^3m(m-1)^2 + 2m^3n(n-1)^2 & \text{if } m \text{ is odd } n \text{ is odd} \\ 2n^3m(2m^2 - 2m + 1) & \text{if } m \text{ is odd } n \text{ is even} \\ 2m^3n(2n^2 - 2n + 1) & \text{if } m \text{ is even } n \text{ is odd} \\ 4m^3n^3 & \text{if } m \text{ is even } n \text{ is even.} \end{cases}$$

$$4. Sz_w(T) = \frac{2m^2(n-1)}{3}(2mn^2 + 2mn - 3m - n^2 - n) + \frac{2n^2(m-1)}{3}(2nm^2 + 2mn - 3n - m^2 - m).$$

$$5. Sz_w(C_{n_1} \square C_{n_2} \square \dots \square C_{n_k}) = \begin{cases} k^2 \prod_{i=1}^k n_i^3 & \text{if each } n_i \text{ is even} \\ k \prod_{i=1}^k n_i^3 \sum_{i=1}^k (1 - \frac{1}{n_i})^2 & \text{if each } n_i \text{ is odd.} \end{cases}$$

If each  $n_i = n$ , then  $Sz_w(C_n^k) = \begin{cases} k^2 n^{3k} & \text{if each } n_i \text{ is even} \\ k^2 (n-1)^2 n^{3k-2} & \text{if each } n_i \text{ is odd.} \end{cases}$

**Example 2.8.** Let  $G = C_{n_1} \square C_{n_2} \square \dots \square C_{n_k}$  and  $H = C_{m_1} \square C_{m_2} \square \dots \square C_{m_r}$ , where  $n_i, 1 \leq i \leq k$  are even and  $m_j, 1 \leq j \leq r$  are odd. Using Theorem 2.2 and the above example we obtain the weighted Szeged index of the graph  $G \square H$ .

$$Sz_w(G \square H) = \left( \prod_{i=1}^k n_i^3 \right) \left( \prod_{j=1}^r m_j^3 \right) \left( k^2 + kr + (k+r) \sum_{i=1}^r r(1 - \frac{1}{m_i})^2 \right).$$

If each  $n_i = n \geq 3$  is even and  $m_j = m \geq 3$  is odd, then  $G = C_n^k$  and  $H = C_m^r$  and  $Sz_w(G \square H) = n^{3k} m^{3r} \left( k^2 + kr + (k+r)r(1 - \frac{1}{m})^2 \right)$ .

**Example 2.9.** Using Theorem 2.3, we obtain the exact weighted Szeged index of the grid graph  $P_{n_1} \square P_{n_2} \square \dots \square P_{n_k}$ .

$$Sz_w(P_{n_1} \square P_{n_2} \square \dots \square P_{n_k}) = \frac{2}{3} \left( \prod_{i=1}^k n_i^3 \right) \left( \sum_{i=1}^k \frac{(n_i-1)(n_i^2+n_i-3)}{n_i^3} + \sum_{i,j=1, i \neq j}^k (1 - \frac{1}{n_i})(1 + \frac{1}{n_i})(1 - \frac{1}{n_j}) \right).$$

If each  $n_i = n$ , then  $Sz_w(P_n^k) = \frac{2k(n-1)n^{3(k-1)}}{3} ((n^2 + n - 3) + (k-1)(n+1))$ .

### 3. Weighted Szeged index of the Corona product of graphs

In this section, we compute the weighted Szeged index of the corona product  $G \circ H$  of the graphs  $G$  and  $H$ . The edge  $a$ -Zagreb index and  $(a, b)$ -Zagreb index of  $G$  are defined, in order, as follows

$$Z_a(G) = \sum_{uv \in E(G)} \left( d_G(u)^a + d_G(v)^a \right)$$

$$Z'_{a,b}(G) = \frac{1}{2} \sum_{uv \in E(G)} \left( d_G(u)^a d_G(v)^b + d_G(u)^b d_G(v)^a \right).$$

It is not hard to see that  $Z_1(G) = M_1(G)$  and  $Z'_{1,1} = M_2(G)$ , where  $M_1(G)$  and  $M_2(G)$  are the first and second Zagreb indices of  $G$ , respectively. The first and second Zagreb indices were first introduced in [15] also see [16].

For our convenience, we partition the edge set of  $G \circ H$  into three sets,  $E_1 = \{e \in E(G \circ H) \mid e \in E(H_i), 1 \leq i \leq n\}$ ,  $E_2 = \{e \in E(G \circ H) \mid e \in E(G)\}$  and  $E_3 = \{e \in E(G \circ H) \mid e = uv, u \in V(H_i), 1 \leq i \leq n, v \in V(G)\}$ . It is easy to see that  $E_1, E_2$  and  $E_3$  is a partition of the edge set of  $G \circ H$  and

also  $|E_1| = |V(G)||E(H)|$ ,  $|E_2| = |E(G)|$  and  $|E_3| = |V(G)||V(H)|$ . Let  $t_e(G)$  denote the number of triangles containing the edge  $e$  in  $G$ . The following lemma is used in the proof of the main theorem of this section.

**Lemma 3.1.** *Let  $G$  and  $H$  be graphs. Then*

- (1)  $|V(G \circ H)| = |V(G)|(1 + |V(H)|)$  and  $|E(G \circ H)| = |E(G)| + |V(G)|(|V(H)| + |E(H)|)$ .
- (2) (i) If  $e = uv \in E_1$ , then  $d_{G \circ H}(u) = d_H(u) + 1$  and  $d_{G \circ H}(v) = d_H(v) + 1$ .
- (ii) If  $e = uv \in E_2$ , then  $d_{G \circ H}(u) = d_G(u) + |V(H)|$  and  $d_{G \circ H}(v) = d_G(v) + |V(H)|$ .
- (iii) If  $e = uv \in E_3$  and  $u \in V(H)$ ,  $v \in V(G)$ , then  $d_{G \circ H}(u) = d_H(u) + 1$  and  $d_{G \circ H}(v) = d_G(v) + |V(H)|$ .

**Theorem 3.2.** *Let  $G$  be connected graph of order  $n$  and size  $p$ . If  $H$  is a graph of order  $m$  and size  $q$ , then  $Sz_w(G \circ H) = 2nZ'_{2,1}(H) - n \sum_{e=uv \in E(H)} t_e(H)(d_H(u) + d_H(v))^2 + n \sum_{e=uv \in E(H)} (d_H(u) + d_H(v))(t_e^2(H) - 2t_e(H)) + 2nM_2(H) + 2n \sum_{e \in E(H)} t_e^2(H) + (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m + 1)q$ .*

**Proof.** As mentioned in the beginning of this section, we partition the edge of  $G \circ H$  into three sets  $E_1, E_2$  and  $E_3$ , where  $E_1 = \{e \in E(G \circ H) | e \in E(H_i), 1 \leq i \leq n\}$ ,  $E_2 = \{e \in E(G \circ H) | e \in E(G)\}$  and  $E_3 = \{e \in E(G \circ H) | e = uv \in, u \in V(H_i), 1 \leq i \leq n, v \in V(G)\}$ .

Let  $e = uv \in E_1$ , if  $x \in V(H)$  is non adjacent to both  $u$  and  $v$  in  $H$ , then  $d_{G \circ H}(x, u) = d_{G \circ H}(x, v) = 2$ . Also if  $x \in V(H)$  is adjacent to both  $u$  and  $v$  in  $H$ , then  $d_{G \circ H}(x, u) = d_{G \circ H}(x, v) = 1$ . For both case  $x$  is an equidistant to  $e$  in  $G \circ H$ . Hence  $n_u(e, G \circ H) = d_H(u) - t_e(H)$  and  $n_v(e, G \circ H) = d_H(v) - t_e(H)$ .

$$\begin{aligned}
 & \sum_{e=uv \in E_1} (d_{G \circ H}(u) + d_{G \circ H}(v))n_u(e, G \circ H)n_v(e, G \circ H) \\
 = & n \sum_{e=uv \in E(H)} (d_H(u) + d_H(v) + 2)(d_H(u) - t_e(H))(d_H(v) - t_e(H)), \text{ by Lemma 3.1} \\
 = & n \sum_{e=uv \in E(H)} (d_H(u)^2d_H(v) + d_H(u)d_H(v)^2) - n \sum_{e=uv \in E(H)} t_e(H)(d_H(u) + d_H(v))^2 \\
 & + n \sum_{e=uv \in E(H)} (d_H(u) + d_H(v))(t_e^2(H) - 2t_e(H)) + 2nM_2(H) + 2n \sum_{e \in E(H)} t_e^2(H) \\
 = & 2nZ'_{2,1}(H) - n \sum_{e=uv \in E(H)} t_e(H)(d_H(u) + d_H(v))^2 \\
 (3.1) \quad & + n \sum_{e=uv \in E(H)} (d_H(u) + d_H(v))(t_e^2(H) - 2t_e(H)) + 2nM_2(H) + 2n \sum_{e \in E(H)} t_e^2(H)
 \end{aligned}$$

Let  $e = uv \in E_2$ , if  $x \in T_G(e; u)$ , then all the vertices of the copy of  $H$  attached to  $x$  are in  $T_{G \circ H}(e; u)$ . Since  $|V(H)| = m$ ,  $n_u(e, G \circ H) = (m + 1)n_u(e, G)$  and  $n_v(e, G \circ H) = (m + 1)n_v(e, G)$ .

$$\begin{aligned}
 & \sum_{e=uv \in E_2} (d_{G \circ H}(u) + d_{G \circ H}(v))n_u(e, G \circ H)n_v(e, G \circ H) \\
 &= \sum_{e=uv \in E(G)} (d_G(u) + d_G(v) + 2m)(m + 1)^2n_u(e, G)n_v(e, G), \text{ by Lemma 3.1} \\
 (3.2) \quad &= (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G).
 \end{aligned}$$

Let  $e = uv \in E_3$ , if  $u_1, u_2, \dots, u_r$  are the vertices adjacent to  $u$  in  $H$ , then  $u_j, j = 1, 2, \dots, r$  is equidistant to  $e$  in  $G \circ H$ . On the other hand every vertex of  $G \circ H$  other than  $u, u_1, u_2, \dots, u_r$  are in  $T_{G \circ H}(e; v)$ . Hence  $n_u(e, G \circ H) = 1$  and  $n_v(e, G \circ H) = |V(G \circ H)| - (d_H(u) + 1)$ .

$$\begin{aligned}
 & \sum_{e=uv \in E_3} (d_{G \circ H}(u) + d_{G \circ H}(v))n_u(e, G \circ H)n_v(e, G \circ H) \\
 &= \sum_{u \in V(H)} \sum_{v \in V(G)} (d_H(u) + 1 + d_G(v) + m)(|V(G \circ H)| - (d_H(u) + 1)), \text{ by Lemma 3.1} \\
 (3.3) \quad &= (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m + 1)q.
 \end{aligned}$$

Now we shall obtain the  $Sz_w(G \circ H)$ . By the definition of  $Sz_w(G \circ H)$ ,

$$\begin{aligned}
 Sz_w(G \circ H) &= \sum_{e \in E(G \circ H)} (d_{G \circ H}(u) + d_{G \circ H}(v))n_u(e, G \circ H)n_v(e, G \circ H) \\
 &= 2nZ'_{2,1}(H) - n \sum_{e=uv \in E(H)} t_e(H)(d_H(u) + d_H(v))^2 + n \sum_{e=uv \in E(H)} (d_H(u) + d_H(v))(t_e^2(H) - 2t_e(H)) \\
 &\quad + 2nM_2(H) + 2n \sum_{e \in E(H)} t_e^2(H) + (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G) - nM_1(H) - 4pq \\
 &\quad - 2n(m + 1)q + (n + nm - 1)(2nq + 2mp + nm^2 + nm), \text{ using (3.1), (3.2) and (3.3)}
 \end{aligned}$$

by partition  $E(G \circ H)$  into  $E_1, E_2$  and  $E_3$  as beginning of the proof.

Using Theorem 3.2, we have the following corollaries.

**Corollary 3.3.** *Let  $G$  be connected graph of order  $n$  and size  $p$ . If  $H$  is a  $r$ -regular graph of order  $m$  and size  $q$ , then  $Sz_w(G \circ H) = 2nr^3q - 4nr(r + 1) \sum_{e \in E(H)} t_e(H) + 2n(r + 1) \sum_{e \in E(H)} t_e^2(H) + 2nM_2(H) + (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m + 1)q$ .*

As for a triangle free graph  $H$ ,  $t_e(H) = 0$ , for every edge  $e \in E(H)$ , we have the following corollary.

**Corollary 3.4.** *Let  $G$  be connected graph of order  $n$  and size  $p$ . If  $H$  is a triangle free graph of order  $m$  and size  $q$ , then  $Sz_w(G \circ H) = 2nZ'_{2,1}(H) + 2nM_2(H) + (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m + 1)q$ .*

**Corollary 3.5.** *Let  $G$  be connected graph of order  $n$  and size  $p$ . If  $H$  is a  $r$ -regular triangle free graph of order  $m$  and size  $q$ , then  $Sz_w(G \circ H) = 2nr^3q + 2nM_2(H) + (m + 1)^2Sz_w(G) + 2m(m + 1)^2Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m + 1)q$ .*



By direct calculations we obtain the first and second Zagreb indices of  $P_n$  and  $C_n$ .  $M_1(C_n) = 4n$ ,  $n \geq 3$ ,  $M_1(P_1) = 0$ ,  $M_1(P_n) = 4n - 6$ ,  $n > 1$  and  $M_1(K_n) = n(n - 1)^2$ .  $M_2(P_n) = 4(n - 2)$  and  $M_2(C_n) = 4n$ .

For a given graph  $G$ , its  $t$ -fold bristled graph  $Brs_t(G)$  is obtained by attaching  $t$  vertices of degree 0 to each vertex of  $G$ . This graph can be represented as the corona product of  $G$  and complement of a complete graph on  $t$  vertices. The  $t$ -fold bristled graph of a given graph is also known as its  $t$ -thorny graph.

**Example 3.6.** Let  $G$  be a graph with  $n$  vertices and  $p$  edges. Then  $Sz_w(G \circ \overline{K}_t) = (t + 1)^2 Sz_w(G) + 2t(t + 1)^2 Sz(G) + (n + nt - 1)(2pt + nt^2 + nt)$ .

From the above formula, the weighted Szeged indices of  $t$ -fold bristled graph of  $P_n$  and  $C_n$  can easily be computed.

$$Sz_w(P_n \circ \overline{K}_t) = \frac{(n-1)(t+1)^2}{3} \left( 2(n^2 + n - 3) + nt(n + 1) \right) + (n + nt - 1) \left( 2t(n - 1) + nt(t + 1) \right).$$

$$Sz_w(C_n \circ \overline{K}_t) = \begin{cases} \frac{n^3(t+1)^2(t+2)}{2} + nt(n + nt - 1)(t + 3), & \text{if } n \text{ is even} \\ \frac{n(n-1)^2(t+1)^2(t+2)}{2} + nt(n + nt - 1)(t + 3), & \text{if } n \text{ is odd.} \end{cases}$$

A special corona graph  $C_n \circ K_1$ , that is, a cycle with pendant vertex which has  $2n$  vertices. This is called *sunlet* graph.

$$Sz_w(C_n \circ K_1) = \begin{cases} 2n(3n^2 + 4n - 2), & \text{if } n \text{ is even} \\ 2n(3n^2 - 2n - 1), & \text{if } n \text{ is odd.} \end{cases}$$

**Example 3.7.** Let  $H$  be a triangle free graph with  $m$  vertices and  $q$  edges. Then  $Sz_w(K_n \circ H) = 2n Z'_{2,1}(H) + 2nM_2(H) - nM_1(H) + n(n - 1)(m + 1)^2(n + m - 1) + (n + nm - 1)(2nq + nm(n - 1) + nm(m + 1)) - 2nq(n + m)$ . In particular,  $Sz_w(K_1 \circ H) = 2 Z'_{2,1}(H) + 2M_2(H) - M_1(H) + m^2(m + 1) - 2q$ .

Star graph  $S_{m+1}$  on  $m + 1$  vertices is the corona product of  $K_1$  and  $\overline{K}_m$ . Fan graph  $F_{m+1}$  and wheel graph  $W_{m+1}$  on  $m + 1$  vertices are also corona product of  $K_1$  and  $P_m$  and  $C_m$ . For the graphs  $P_m$  and  $C_m$ ,  $Z'_{2,1}(P_m) = 8m - 18$  and  $Z'_{2,1}(C_m) = 8m$ . From the above formula the weighted Szeged indices of these graphs are obtained.

$$Sz_w(K_1 \circ \overline{K}_m) = m^2(m + 1).$$

$$Sz_w(K_1 \circ P_m) = \begin{cases} m(m - 1)^2, & \text{if } m = 2 \\ m^3 + m^2 + 18m - 44, & \text{if } m \geq 3. \end{cases}$$

$$Sz_w(K_1 \circ C_m) = \begin{cases} m^3 + m^2 + 18m - 54, & \text{if } m = 3 \\ m^3 + m^2 + 18m, & \text{if } m > 3. \end{cases}$$

Corona product sometimes appear in chemical literature as plerographs of the usual hydrogen-suppressed molecular graphs knowns as kenographs, see [20] for definitions and more information. For example, for a given graph  $H$  the graph  $K_2 \circ H$  is called the *bottleneck* graph of  $H$ . The Weighted Szeged index of this graph can easily be obtained from Example 3.7.

**Example 3.8.** Let  $H$  be a triangle free graph with  $m$  vertices and  $q$  edges. Then  $Sz_w(K_2 \circ H) = 4Z'_{2,1}(H) + 4M_2(H) - 2M_1(H) + 2(3m^3 + 8m^2 + 5m + 1) - 2q(m + 2)$ . In particular, the Weighted Szeged index of the bottleneck graph of  $P_m$  is equal to  $Sz_w(K_2 \circ P_m) = 6m^3 + 146m^2 - 32m - 86$ .

Let  $\{G_i\}_{i=1}^n$  be a set of finite pairwise disjoint graphs with  $v_i \in V(G_i)$ . The bridge graph

$$B\{G_1, G_2, \dots, G_n; v_1, v_2, \dots, v_n\}$$

of  $\{G_i\}_{i=1}^n$  with respect to the vertices  $\{v_i\}_{i=1}^n$  is the graph obtained from the graphs  $G_1, G_2, \dots, G_n$  by connecting the vertices  $v_i$  and  $v_{i+1}$  by an edge for all  $i = 1, 2, \dots, n - 1$ .

We define  $G_n(H, v) = B\{\underbrace{H, H, \dots, H}_{n \text{ times}}; \underbrace{v, v, \dots, v}_{n \text{ times}}\}$ , ( $n$  times) which is the special case of bridge graph. For example, let  $P_n$  be the path on  $n$  vertices  $v_1, v_2, \dots, v_n$ , define  $B_n = G_n(P_3, v_2)$ , in particular, it is a Polyethene when  $n = 4$ . As another example, let  $C_k$  be the cycle with  $k$  vertices and define  $T_n = G_n(C_k, v_1)$  (when  $k = 3$  and  $n = 5$ ). As a final example, define the bridge graph  $J_{n,m+1} = G_n(W_{m+1}, v_1)$ , where  $W_{m+1}$  is the Wheel graph on  $m + 1$  vertices  $v_1, v_2, \dots, v_{m+1}$  such that  $deg(v_1) = m$  and  $deg(v_i) = 3$ ,  $i = 1, 2, \dots, m + 1$ . By the definition of corona product,  $B_n = P_n \circ \overline{K}_2$ ,  $T_{n,3} = P_n \circ K_2$  and  $J_{n,m+1} = P_n \circ C_m$ .

**Example 3.9.** Using Theorem 3.2, we obtain the weighted Szeged indices of the following graphs.

1.  $Sz_w(B_n) = 2(6n^3 + 15n^2 - 26n + 11)$ .
2.  $Sz_w(T_{n,3}) = 2(6n^3 + 18n^2 - 31n + 13)$ .
3.  $Sz_w(J_{n,m+1}) = \frac{(m+1)^2(n-1)(n(n+1)(m+2)-6)}{3} + m(5n^2 + 6n^2m + n^2m^2 - 3mn + 5n + 6)$ .

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